*Systems of Ternariants that are Algebraically Complete.**

BY A. R. FORSYTH, M. A., F. R. S., *Fellow of Trinity College, Cambridge.*

The present memoir is divided into three parts; it deals with the theory of the algebraically independent concomitants of ternary quantics, taking as the starting point the six linear partial differential equations of the first order satisfied by them.

In Part I it is proved by means of these differential equations that, if any concomitant be expanded in powers of x_1, x_2, x_3—the ordinary (point) variables—and of u_1, u_2, u_3—the contragredient (line) variables—it is completely determinate if its leading coefficient be known; i. e. the coefficient of the term involving the highest power of x_1 and the highest power of u_1; and that every such leading coefficient is a simultaneous solution of two linear partial differential equations of the first order.

Hence if all the independent solutions of these two equations be obtained, all the independent concomitants are given—the independence considered being algebraical and not asyzygetic. For it follows from the theory of such differential equations that every solution can be expressed in terms of a finite number of such solutions, and therefore, on account of the uniqueness of the concomitant derived from a leading coefficient, that every concomitant can be expressed in terms of a finite number of independent concomitants.

Again, it follows from the forms of the two characteristic differential equations that every leading coefficient is a simultaneous concomitant of certain binary quantics constructed from the original ternary quantic, two of the coeffi-

* It has proved convenient to use the word "ternariants" as a generic term for concomitants of ternary quantics, instead of giving it the signification which Prof. Sylvester, to whom so much of the nomenclature of the theory of forms is due, proposed (*Amer. Journ. of Math.*, Vol. V, p. 81) to give to it, viz. the leading coefficients of those concomitants.

cients of which are their variables and the rest of the coefficients of which are their coefficients. There is thus a corresponding theory of these simultaneous binariants, the forms of which are of the utmost importance in the deduction of the leading coefficients of ternariants.

In Part II the foregoing general theory is applied to a number of cases of uniternary quantics, the two characteristic equations being solved in each case to give the independent simultaneous solutions as leading coefficients. It appears that in the case of the quadratic there are three ternariants of the nature indicated (they are shown to be equivalent to the three asyzygetic ternariants in the ordinary theory of the quadratic); in the case of the cubic there are seven, and in the case of the quartic there are twelve such ternariants. The leading coefficients of each of these are explicitly given for the most general form of quantic; and by one or other of two methods, viz. by means of the differential operators which give the development of the quantic, or by obtaining the symbolical expression of the ternariant with a given leading coefficient, the order in the point-variables and the class in the line-variables are derived. The three essential elements of each ternariant—its source, its order, and its class—being thus known, the full development for tabulation purposes is then only a question of differential operators, or of the realization of symbolical expressions.

Some illustrations are given in connection with the cubic and the quartic, wherein some of the well-known irreducible concomitants are algebraically expressed in terms of members of the fundamental sets. There is thus afforded an opportunity of comparing the functions which are algebraically independent with those which are only asyzygetically independent. Speaking generally, the result of the comparison is that for algebraical independence the order and the class are both much higher than for asyzygetic independence, so that in the case of complete tabulation many more terms in variables would occur; but that the leading (and so all the other) coefficients are much simpler in form (that is, contain fewer terms and are of lower degrees) than the leading coefficients of the asyzygetic ternariants.

The theory is then applied to the unipartite ternary quantic of order n, with the result that its fundamental system contains $\frac{1}{2}(n+4)(n-1)$ algebraically independent concomitants in terms of which every concomitant can be expressed. The forms, sometimes explicit, sometimes symbolical in the notation of only binary quantics, are given for these leading coefficients; the order and

the class of each ternariant thus determined are obtained from the symbolical expressions.

The case of two simultaneous ternary quadratics is next discussed, and it appears that there are nine ternariants in the fundamental system. Some subsidiary ternariants (subsidiary, that is, from the point of view of the fundamental system) are obtained; and the 20 ternariants which constitute the asyzygetic system due to Gordan, are expressed in terms of the foregoing system.

Lastly, the fundamental system of three simultaneous ternary quadratics is obtained.

In Part III the general theory is applied to a number of cases of biternary quantics; the special cases treated at any length being the systems of the lineo-linear, the quadrato-linear, the cubo-linear, the quadrato-quadratic, the cubo-cubic, and the system of two lineo-linear quantics. Finally, the fundamental system of the biternary n^0m^{ic} is obtained, containing

$$\tfrac{1}{4}(n+1)(n+2)(m+1)(m+2)-3$$

ternariants, of which the greatest number are, for this most general case, given in symbolic form (the equivalent of solutions of the two characteristic equations), and for each of which the order and the class are derived from that symbolical form. In some cases of the last, ternariants syzygetically equivalent, save for a power of u_x, to members of the fundamental system, are given.

There is an immense mass of literature on the subject of ternary forms; but, so far as I can see, it deals with asyzygetic ternariants which are more difficult to treat, and have been given in full systems only for the uni-ternary quadratic, the uni-ternary cubic, and a system of two uni-ternary quadratics, for a special form of uni-ternary quartic and for a biternary lineo-linear quantic; and part of the system has been given for the general uni-ternary quartic.

Some of the principal papers dealing with the theory of the cubic are given in Cayley's memoir.* Gordan's memoirs on the special form of the ternary quartic occur in the 17th and 20th volumes of the Mathematische Annalen; the concomitants of lowest degrees for the general quartic are given in a memoir by Maisano (quoted in §42); and several of the concomitants are tabulated in full

* *Amer. Journ. of Math.*, Vol. IV, 1881, On the 34 Concomitants of the Ternary Cubic, pp. 1–15.

in a memoir by Joubert.* In connection with the general theory, a memoir by Sylvester† may be studied; and also a recent memoir by Brioschi.‡ In a memoir by Bernardi,§ the characteristic differential equations satisfied by concomitants of uni-ternary concomitants are given, and the Hessian and the cubinvariant of the quartic are there calculated in full.

In regard to biternary forms in sets of contragredient variables, the most important memoir is by Clebsch and Gordan, quoted in the note to §60; in that note a fairly complete list of the more important papers dealing with bipartite forms is given, though most of them discuss the transformation of bilinear forms.||

PART I.

GENERAL THEORY.

The Differential Equations Satisfied by the Concomitants.

1. In the discussion of the concomitants of ternary quantics, the ordinary (point) variables will be denoted by x_1, x_2, x_3; the contragredient (line) variables by u_1, u_2, u_3. The general uni-ternary quantic of order n may be represented either in the symbolical form

$$f = a_x^n = (a_1x_1 + a_2x_2 + a_3x_3)^n,$$

or in the reduced non-symbolical form

$$f = (\ldots\ldots, a_{rst}, \ldots\ldots)(x_1, x_2, x_3)^n,$$

where the coefficient of $a_{rst}x_1^rx_2^sx_3^t$ is the multinomial coefficient $n! \div (r!\ s!\ t!)$ with the condition $r + s + t = n$.

Every concomitant ϕ, of general order m in the point variables and general class p in the line variables, satisfies** the six characteristic linear partial differential equations

---

* "Sur la théorie algébrique des formes homogènes du quatrième degré à trois indéterminées." Comptes Rendus, t. LVI (1863), pp. 1045–1048, 1088–1091, 1123–1126.

† Comptes Rendus, t. LXXXIV (1877), pp. 1359–1361, 1427–1430.

‡ "Studi sulle forme ternarie," Ann. di Mat., t. XV (1888), pp. 235–252.

§ Batt. Giorn., t. XIX (1881), pp. 136–150, 258–293.

|| A short table of contents of this paper is given at the end.

** "On the Differential Equations satisfied by Concomitants of Quantics," Proc. Lond. Math. Soc., Vol. XIX (1888), p. 41.

$$\left.\begin{aligned}
x_1\frac{\partial\phi}{\partial x_2}-u_2\frac{\partial\phi}{\partial u_1}&=\Sigma\Sigma r a_{r-1,\, s+1,\, t}\frac{\partial\phi}{\partial a_{rst}}=D_3\phi\\
x_2\frac{\partial\phi}{\partial x_3}-u_3\frac{\partial\phi}{\partial u_2}&=\Sigma\Sigma s a_{r,\, s-1,\, t+1}\frac{\partial\phi}{\partial a_{rst}}=D_1\phi\\
x_3\frac{\partial\phi}{\partial x_1}-u_1\frac{\partial\phi}{\partial u_3}&=\Sigma\Sigma t a_{r+1,\, s,\, t-1}\frac{\partial\phi}{\partial a_{rst}}=D_2\phi\\
x_2\frac{\partial\phi}{\partial x_1}-u_1\frac{\partial\phi}{\partial u_2}&=\Sigma\Sigma s a_{r+1,\, s-1,\, t}\frac{\partial\phi}{\partial a_{rst}}=D_4\phi\\
x_3\frac{\partial\phi}{\partial x_2}-u_2\frac{\partial\phi}{\partial u_3}&=\Sigma\Sigma t a_{r,\, s+1,\, t-1}\frac{\partial\phi}{\partial a_{rst}}=D_6\phi\\
x_1\frac{\partial\phi}{\partial x_3}-u_3\frac{\partial\phi}{\partial u_1}&=\Sigma\Sigma r a_{r-1,\, s,\, t+1}\frac{\partial\phi}{\partial a_{rst}}=D_5\phi
\end{aligned}\right\}\quad(1),$$

where the summations in the literal operators extend to all distinct integral combinations of r, s, t subject to the relation $r+s+t=n$.

2. The 15 Jacobian conditions, which must be satisfied in order that these equations may have common solutions, resolve themselves into three classes. The first class consists of 6 equations satisfied identically, in virtue of the relations

$$[D_3, D_6]=0,\quad [D_1, D_5]=0,\quad [D_2, D_4]=0,$$
$$[D_3, D_5]=0,\quad [D_1, D_4]=0,\quad [D_2, D_6]=0.$$

The second class consists of six equations satisfied by means of some one of the equations (1), in virtue of the relations

$$[D_1, D_2]=D_4,\quad [D_2, D_3]=D_6,\quad [D_3, D_1]=D_5,$$
$$[D_4, D_5]=D_1,\quad [D_5, D_6]=D_3,\quad [D_6, D_4]=D_2.$$

The third class consists of three quasi-homogeneous equations, viz.

$$\left.\begin{aligned}
D_9\phi=[D_3, D_4]\,\phi&=x_1\frac{\partial\phi}{\partial x_1}-x_2\frac{\partial\phi}{\partial x_2}-u_1\frac{\partial\phi}{\partial u_1}+u_2\frac{\partial\phi}{\partial u_2}\\
D_8\phi=[D_2, D_5]\,\phi&=x_3\frac{\partial\phi}{\partial x_3}-x_1\frac{\partial\phi}{\partial x_1}-u_3\frac{\partial\phi}{\partial u_3}+u_1\frac{\partial\phi}{\partial u_1}\\
D_7\phi=[D_1, D_6]\,\phi&=x_2\frac{\partial\phi}{\partial x_2}-x_3\frac{\partial\phi}{\partial x_3}-u_2\frac{\partial\phi}{\partial u_2}+u_3\frac{\partial\phi}{\partial u_3}
\end{aligned}\right\}\quad(2),$$

where the operators D_7, D_8, D_9 are

$$\left.\begin{aligned}
D_7&=\Sigma\Sigma\,(s-t)\,a_{rst}\frac{\partial}{\partial a_{rst}}\\
D_8&=\Sigma\Sigma\,(t-r)\,a_{rst}\frac{\partial}{\partial a_{rst}}\\
D_9&=\Sigma\Sigma\,(r-s)\,a_{rst}\frac{\partial}{\partial a_{rst}}
\end{aligned}\right\}\quad(3).$$

And the further Jacobian conditions, due to the introduction of the new equations (2), are found to be satisfied, in virtue of relations of the form

$$\begin{array}{lll} [D_1, D_9] = D_1, & [D_5, D_9] = -D_5, & [D_3, D_9] = -2D_3, \\ [D_2, D_9] = D_2, & [D_6, D_9] = -D_6, & [D_4, D_9] = 2D_4, \end{array}$$

with similar relations for the other two operators D_7 and D_8.

3. But the equations in the foregoing system, sufficient for the derivation of every partial differential equation satisfied by a concomitant, are not independent of one another.

Let any equation be written in the form $P'\phi = P\phi$, where P is the literal operator and P' is the variable operator; then any one of the nine operators P is commutative with any one of the nine operators P'. Consider, then, any function Φ which satisfies two of the equations, say

$$P'\phi = P\phi, \quad Q'\phi = Q\phi;$$

then we have

$$\begin{aligned}(PQ - QP)\phi &= (PQ' - QP')\phi, \\ &= (Q'P - P'Q)\phi, \\ &= (Q'P' - P'Q')\phi.\end{aligned}$$

If the operators P and Q be commutative, then the equation just obtained is an evanescent identity; but if they be not commutative, the equation is not an identity and yet it is different in form from either of the two from which it is derived. It is thus a new equation,

$$R\phi = R'\phi,$$

satisfied in virtue of the two former equations.

When this process is applied to the foregoing system of nine equations, it appears that they can be reduced to a system of three equations, independent of one another, by means of the following sets of relations:

$$\left.\begin{aligned} D'_1 &= D'_4D'_5 - D'_5D'_4 \\ D'_2 &= D'_6D'_4 - D'_4D'_6 \\ D'_3 &= D'_5D'_6 - D'_6D'_5 \\ D'_4 &= D'_1D'_2 - D'_2D'_1 \\ D'_5 &= D'_3D'_1 - D'_1D'_3 \\ D'_6 &= D'_2D'_3 - D'_3D'_2 \\ D'_7 &= D'_1D'_6 - D'_6D'_1 \\ D'_8 &= D'_2D'_5 - D'_5D'_2 \\ D'_9 &= D'_3D'_4 - D'_4D'_3 \end{aligned}\right\}; \qquad \left.\begin{aligned} D_1 &= D_5D_4 - D_4D_5 \\ D_2 &= D_4D_6 - D_6D_4 \\ D_3 &= D_6D_5 - D_5D_6 \\ D_4 &= D_2D_1 - D_1D_2 \\ D_5 &= D_1D_3 - D_3D_1 \\ D_6 &= D_3D_2 - D_2D_3 \\ D_7 &= D_6D_1 - D_1D_6 \\ D_8 &= D_5D_2 - D_2D_5 \\ D_9 &= D_4D_3 - D_3D_4 \end{aligned}\right\}.$$

Thus, for instance, the nine equations can be made to depend on the set

$$D_1\phi = D_1'\phi\,; \quad D_2\phi = D_2'\phi\,; \quad D_3\phi = D_3'\phi\,.$$

But it will be convenient to retain the whole system, for it includes the full aggregate of equations which are similar to one another in form.

All combinations, other than those which occur in the foregoing set of combinations, are connected with operators which are lineo-commutative; e. g. $D_1D_4 - D_4D_1 = 0$. And it may be remarked that the operators in the foregoing set of combinations are quadrato-commutative, according to laws of the form

$$D_2D_1^2 - 2D_1D_2D_1 + D_1^2D_2 = 0\,,$$
$$D_1D_2^2 - 2D_2D_1D_2 + D_2^2D_1 = 0\,.$$

4. It is convenient to assign certain weights to the various quantities which enter. We assign the weight zero to x_3, unity to x_2, and ρ (unspecified but, if desirable, different from zero or unity) to x_1. Since f and u_x $(= u_1x_1 + u_2x_2 + u_3x_3)$ must be isobaric, we assign to a_1 and u_1 the weight zero, to a_2 and u_2 the weight $\rho - 1$, and to a_3 and u_3 the weight ρ.* Then the weight of the coefficient $a_{r,\,s,\,t}$ is $s(\rho - 1) + t\rho$; and the following changes are caused on any isobaric function by the operators:

The operator D_1 increases and D_6 decreases the weight by 1,
" " D_3 " " D_4 " " " " $\rho - 1$,
" " D_5 " " D_2 " " " " ρ.

5. Now suppose the concomitant Φ expanded in powers of the point variables, in which its order is m; this expansion is of the form

$$\Phi = x_1^m\Phi_{0,\,0} + \ldots\ldots + \frac{x_1^{m-r-s}x_2^rx_3^s}{r!\,s!}\Phi_{r,\,s} + \ldots\ldots \tag{4}$$

When this expression is substituted in the six fundamental characteristic equations (1), the result is in each case an identity; hence, by comparison of the coefficients of the various x-combinations, we have the relations

$$\left.\begin{aligned} D_3\Phi_{r,\,s} + u_2\frac{\partial\Phi_{r,\,s}}{\partial u_1} &= \Phi_{r+1,\,s} \\ D_5\Phi_{r,\,s} + u_3\frac{\partial\Phi_{r,\,s}}{\partial u_1} &= \Phi_{r,\,s+1} \end{aligned}\right\} \tag{5}$$

*The weights σ, $\sigma + \rho - 1$, $\sigma + \rho$ (σ unspecified) might be assigned to u_1, u_2, u_3; but the foregoing assignation is equally efficient for the purpose of obtaining the difference in weights of the variable parts of two terms of a concomitant.

$$\left.\begin{aligned} D_1\Phi_{r,s}+u_3\frac{\partial\Phi_{r,s}}{\partial u_2}&=r\Phi_{r-1,s+1}\\ D_6\Phi_{r,s}+u_2\frac{\partial\Phi_{r,s}}{\partial u_3}&=s\Phi_{r+1,s-1}\\ D_2\Phi_{r,s}+u_1\frac{\partial\Phi_{r,s}}{\partial u_3}&=(m-r-s+1)s\Phi_{r,s-1}\\ D_4\Phi_{r,s}+u_1\frac{\partial\Phi_{r,s}}{\partial u_2}&=(m-r-s+1)r\Phi_{r-1,s}\end{aligned}\right\}.$$

From the first pair of these, viz. (5), it follows that if Φ_{00} and m be known, the full expansion of the covariant can be obtained merely by processes of differentiation, for the two equations give the relation

$$\Phi_{r,s}=\left(D_3+u_2\frac{\partial}{\partial u_1}\right)^r\left(D_5+u_3\frac{\partial}{\partial u_1}\right)^s\Phi_{0,0} \qquad \text{(I)}.$$

And from the remaining four it follows that $\Phi_{0,0}\,(=P)$ satisfies the four equations

$$\left.\begin{aligned} D_1P+u_3\frac{\partial P}{\partial u_2}=0,\quad D_2P+u_1\frac{\partial P}{\partial u_3}=0\\ D_6P+u_2\frac{\partial P}{\partial u_3}=0,\quad D_4P+u_1\frac{\partial P}{\partial u_2}=0\end{aligned}\right\} \qquad (6).$$

6. One immediate inference as to the isobaric character of a concomitant can be derived; for if P be assumed isobaric and of weight ε, then the weight of $\Phi_{r,s}$ is, on account of the effect of the operators D_3 and D_5, equal to $\varepsilon+r(\rho-1)+s\rho$, and therefore the weight of the term $x_1^{m-r-s}x_2^r x_3^s\Phi_{r,s}$ is

$$(m-r-s)\,\rho+r+\varepsilon+\varepsilon\,(\rho-1)+s\rho,$$

that is, it is $m\rho+\varepsilon$, and is therefore the same for every term.

7. Let $\Phi_{00}\,(=P)$, which in general is a function of u_1, u_2, u_3 of class p, be expanded in powers of these variables in the form

$$P=u_1^p\psi_{0,0}+\ldots\ldots+\frac{u_1^{p-q-t}u_2^q u_3^t}{q!\,t!}\psi_{q,t}+\ldots\ldots \qquad (7),$$

in which the quantities ψ involve only the coefficients of the quantic. Then proceeding as before, a comparison of the various combinations of the variables in the equations (6) after the substitution of P leads to the relations

$$\left.\begin{aligned} D_1\psi_{q,t}+s\psi_{q+1,t-1}&=0\\ D_6\psi_{q,t}+r\psi_{q-1,t+1}&=0\end{aligned}\right\}$$
$$\left.\begin{aligned} D_2\psi_{q,t}+\ \psi_{q,t+1}&=0\\ D_4\psi_{q,t}+\ \psi_{q+1,t}&=0\end{aligned}\right\} \qquad (8).$$

From the last pair of these, viz. (8), it follows that if $\psi_{0,\,0}$ be known and also p, the full expansion of P can be obtained merely by processes of differentiation; for the two equations give the relation

$$\psi_{q,\,t} = (-1)^{q+t} D_4^q D_2^t \psi_{0,\,0} \tag{II}$$

But this relation shows that if $\psi_{0,\,0}$ be known, the value of p can be derived from it. For the term involving the highest power of u_3 has, save as to a numerical factor, the coefficient $\psi_{0,\,p}$; and a coefficient $\psi_{0,\,p+1}$ is necessarily zero. Hence

and similarly
$$\left.\begin{aligned} D_2^{p+1}\psi_{0,\,0} &= 0 \\ D_4^{p+1}\psi_{0,\,0} &= 0 \end{aligned}\right\} \tag{9}$$

The value of p can thus be obtained when $\psi_{0,\,0}$ is given by operating with D_2 or D_4 a number of times in succession; the value is evidently less by unity than the number of times first necessary to give a zero result.

From the former pair of equations it follows that $\psi_{0,\,0}$ satisfies the two equations

$$D_1\psi_{0,\,0} = 0, \quad D_6\psi_{0,\,0} = 0 \tag{10}$$

8. Before proceeding to discuss the effect of the subsidiary characteristic equations (2), it is desirable to reconsider the main equations (1).

The preceding results have been deduced on the supposition that the concomitant Φ is most conveniently arranged initially in powers of the point-variables. But if we take an alternative initial expression in powers of the line-variables—a necessity in the case of pure contravariants—in the form

$$\Phi = u_1^p \chi_{0,\,0} + \ldots + (-1)^{r+s} \frac{u_1^{p-r-s} u_2^r u_3^s}{r!\,s!} \chi_{r,\,s} + \ldots \tag{4'}$$

and substitute in the original equations, then similar analysis gives the two results: (i) that $\chi_{r,\,s}$ is determined from $\chi_{0,\,0}$ by the relation

$$\chi_{r,\,s} = \left(D_2 - x_3 \frac{\partial}{\partial x_1}\right)^r \left(D_4 - x_2 \frac{\partial}{\partial x_1}\right)^s \chi_{0,\,0} \tag{III}$$

and (ii) that $\chi_{0,\,0}\,(= Q)$ satisfies the equations

$$\left.\begin{aligned} D_3 Q &= x_1 \frac{\partial Q}{\partial x_2}, & D_1 Q &= x_2 \frac{\partial Q}{\partial x_3} \\ D_5 Q &= x_1 \frac{\partial Q}{\partial x_3}, & D_6 Q &= x_3 \frac{\partial Q}{\partial x_2} \end{aligned}\right\} \tag{6'}$$

And when Q, a function of the point-variables, is expanded in the form

$$Q = x_1^m \theta_{0,0} + \ldots + \frac{x_1^{m-q-t} x_2^q x_3^t}{q!\, t!} \theta_{q,t} + \ldots \tag{7'}$$

(with the evident relation $\theta_{0,0} = \psi_{0,0}$, each being the coefficient of $x_1^m u_1^p$ in Φ), then, by means of the equations (6′) it follows that

$$\theta_{q,t} = D_3^q D_5^t \theta_{0,0} = D_3^q D_5^t \psi_{0,0} \tag{IV};$$

that m can be determined by either of the equations

$$\left.\begin{array}{l} D_3^{m+1} \psi_{0,0} = 0 \\ D_5^{m+1} \psi_{0,0} = 0 \end{array}\right\} \tag{11},$$

and that $\theta_{0,0}\,(= \psi_{0,0})$ satisfies the same equations as before, viz.

$$D_1 \psi_{0,0} = 0, \quad D_6 \psi_{0,0} = 0 \tag{10'}.$$

9. Hence it follows that a concomitant is uniquely determined by the coefficient of its leading term; for by (9) and (11) its order and class are deducible from that leading coefficient, and either by means of (I) and (II) or by means of (III) and (IV) the full expansion can be obtained.

10. The determination of a concomitant thus resolves itself into the determination of the leading coefficient of that concomitant. We have already seen that it must satisfy the two equations (10); there remains the consideration of the effect of the subsidiary equations (2) on the leading coefficient. When the expanded form of the concomitant Φ is substituted, then so far as $\Phi_{0,0}$ is concerned we have the relations

$$D_9 \psi_{0,0} = (m - p)\, \psi_{0,0} = -D_8 \psi_{0,0}; \quad D_7 \psi_{0,0} = 0,$$

which, on account of the forms of the operators D_7, D_8, D_9 are equivalent to

$$\left.\begin{array}{l} D_7 \psi_{0,0} = 0 \\ D_9 \psi_{0,0} = (m - p)\, \psi_{0,0} \end{array}\right\} \tag{12}.$$

Hence it follows that $\psi_{0,0}$ satisfies the two characteristic equations (10) and the two quasi-homogeneous equations (12).

11. As in the corresponding theorem in binary quantics, it follows that every function, derived by the complete set of equations through an isobaric solution $\psi_{0,0}$ of the determining differential equations (10) and (12), is a concomitant.

12. Two remarks remain to be made: one is that the operators used in

deducing the expansion of the concomitant are, in pairs, applicable in any order, viz. $D_3 + u_2 \dfrac{\partial}{\partial u_1}$ and $D_5 + u_3 \dfrac{\partial}{\partial u_1}$, D_3 and D_5, $D_4 - x_2 \dfrac{\partial}{\partial x_1}$ and $D_2 - x_3 \dfrac{\partial}{\partial x_1}$, D_4 and D_2. The other is that the expansion has been derived in descending powers of x_1 and u_1, while it might equally well have been so derived in regard to x_2 and u_2, or x_3 and u_3; and thus the leading coefficients of concomitants (which are not invariants) will be specially associated with coefficients of terms involving high powers of x_1 in the quantic, as later coefficients will be specially associated with coefficients of terms involving high powers of x_2 and of x_3.

13. The general inferences as to the equations which determine the various classes of concomitants are as follow:

The leading coefficients ψ of all *mixed concomitants* (Zwischenformen) satisfy and are determined by the characteristic equations

$$D_1\psi = 0, \quad D_6\psi = 0,$$

and, as will appear subsequently, every adopted solution of these two equations identically satisfies the subsidiary equations

$$D_7\psi = 0, \quad D_9\psi = (m - p)\,\psi,$$

where m is the order of the mixed concomitant in the point-variables and p is its class in the line-variables. The integers m and p are determinable from equations (9) and (11); and the full development of the concomitant is given by equations (I) and (II), or by (III) and (IV).

The leading coefficients of all *pure contravariants* (Zugehörige Formen) satisfy the characteristic equations

$$D_1\psi = 0, \quad D_6\psi = 0, \quad D_3\psi = 0, \quad D_5\psi = 0;$$

they satisfy identically the subsidiary equations

$$D_7\psi = 0, \quad D_9\psi = -p\psi,$$

where p is the class of the contravariant.

The leading coefficients of all *pure covariants* satisfy the characteristic equations

$$D_1\psi = 0, \quad D_6\psi = 0, \quad D_2\psi = 0, \quad D_4\psi = 0;$$

they satisfy identically the subsidiary equations

$$D_7\psi = 0, \quad D_9\psi = m\psi,$$

where m is the order of the covariant.

All *invariants* satisfy the characteristic equations

$$D_1\psi = 0, \quad D_6\psi = 0, \quad D_2\psi = 0, \quad D_4\psi = 0, \quad D_3\psi = 0, \quad D_5\psi = 0;$$

and they satisfy identically the subsidiary equations

$$D_7\psi = 0, \quad D_9\psi = 0.$$

14. It thus appears that from the point of view of analytical derivation of the concomitants, it is in every case necessary to obtain the common solutions of the two equations $D_1\psi = 0 = D_6\psi$, and that from every common solution the corresponding order and class can be deduced. Whenever either the order or the class, or both the order and the class, may happen to be zero, two additional equations for each zero are satisfied by the common solution of $D_1\psi = 0 = D_6\psi$.

It is not, however, necessary to determine both m and p, the order and the class, of a mixed concomitant by the processes indicated. Since $x_1^m u_1^p \psi$ is a term of the concomitant, so also $\pm$ is $x_2^m u_2^p \psi'$, where ψ' is the value of ψ when corresponding coefficients of the quantic are interchanged by the substitution $x_1 = X_2$, $x_2 = X_1$, $x_3 = X_3$. If, then, W be the weight of ψ and W' that of ψ', we have from the isobarism of the concomitant

$$m\rho + W = m + p(\rho - 1) + W',$$

so that

$$m - p = \frac{W' - W}{\rho - 1},$$

a quantity thus determinable by mere inspection. No further relation to determine m and p, other than an equivalent of this relation, can be obtained by such interchanges and substitutions. In using the relation, moreover, it is sufficient to obtain W and W' from any—the simplest—term in ψ.

In order, then, to determine by this method the quantities m and p to be necessarily associated with the concomitant determinable by a given solution ψ, the first step will be to determine the value of $m - p$; the second will naturally be to determine the smaller of the two quantities (should they be unequal) by the equations (9) or (11).

There is, however, another method of proceeding which is much more rapid for the determination of m and p, though less advantageous for tabulation purposes. It is a consequence of the theorem that every ternariant can be represented symbolically, that a leading coefficient is sufficient to determine the ternariant uniquely; if, therefore, a leading coefficient be given, the most rapid

method of obtaining an expression is to change that leading coefficient so that it may be constituted solely by the umbral elements of the original quantic or quantics, and then to complete, by means of the variables, the various factors of that umbral form according to the laws that govern symbolical expressions. Thus for instance an umbral factor a_1 would be completed into a_x, an umbral factor $b_2c_3 - b_3c_2$ into (bcu), and so on.

15. Further, since each of the characteristic equations is linear and partial of the first order, there will be for each quantic a definite number M of algebraically independent solutions of $D_1\psi = 0 = D_6\psi$; and it is a consequence of the theory of such equations that every solution can be expressed as a function of these M solutions. It has been seen that each such isobaric, homogeneous solution determines a concomitant, and therefore for every quantic there is a definite number M of concomitants algebraically independent of one another, such that any concomitant of that quantic can be expressed in terms of those M concomitants, and of the universal concomitant $u_x = u_1x_1 + u_2x_2 + u_3x_3$.

Such a system of concomitants is not unique; it may be replaced by an algebraically equivalent system, containing necessarily the same number of independent concomitants. And the independence is not merely syzygetic, it is an algebraical independence.

16. In the matter of notation, it is desirable to have the coefficients of the quantics so chosen as to render the analytical forms of the characteristic differential equations as simple as possible. Thus the ternary quadratic is taken in the form

$$\begin{aligned} & a_0x_1^2 + 2x_3a_1x_1 + x_3^2a_2; \\ + \; & 2b_0x_1x_2 + 2x_3b_1x_2 \\ + \; & c_0x_2^2; \end{aligned}$$

the ternary cubic in the form

$$\begin{aligned} & a_0x_1^3 + 3x_3a_1x_1^2 + 3x_3^2a_2x_1 + x_3^3a_3; \\ + \; & 3b_0x_1^2x_2 + 3x_32b_1x_1x_2 + 3x_3^2b_2x_2 \\ + \; & 3c_0x_1x_2^2 + 3x_3c_1x_2^2 \\ + \; & d_0x_2^3, \end{aligned}$$

and the arrangement of the coefficients for the general quantic can evidently be made to follow the same law.

With this notation the six literal operators in equations are: first, the two

the simultaneous solutions of which have to be obtained are

$$\left.\begin{aligned} D_1 &= a_1\frac{\partial}{\partial b_0}+a_2\frac{\partial}{\partial b_1}+a_3\frac{\partial}{\partial b_2}+\ldots+2\left(b_1\frac{\partial}{\partial c_0}+b_2\frac{\partial}{\partial c_1}+\ldots\right)+3\left(c_1\frac{\partial}{\partial d_0}+\ldots\right) \\ D_6 &= b_0\frac{\partial}{\partial a_1}+c_0\frac{\partial}{\partial b_1}+d_0\frac{\partial}{\partial c_1}+\ldots+2\left(b_1\frac{\partial}{\partial a_2}+c_1\frac{\partial}{\partial b_2}+\ldots\right)+3\left(b_2\frac{\partial}{\partial a_3}+\ldots\right) \end{aligned}\right\};$$

second, the two which serve to give the development of the concomitant in powers of x and to determine m are

$$\left.\begin{aligned} D_3 = nb_0\frac{\partial}{\partial a_0} &+ (n-1)\,c_0\frac{\partial}{\partial b_0}+(n-2)\,d_0\frac{\partial}{\partial c_0}+\ldots. \\ &+ (n-1)\,b_1\frac{\partial}{\partial a_1}+(n-2)\,c_1\frac{\partial}{\partial b_1}+\ldots.+(n-2)\,b_2\frac{\partial}{\partial a_2}+\ldots. \\ D_5 = na_1\frac{\partial}{\partial a_0} &+ (n-1)\,a_2\frac{\partial}{\partial a_1}+(n-2)\,a_3\frac{\partial}{\partial a_2}+\ldots. \\ &+ (n-1)\,b_1\frac{\partial}{\partial b_0}+(n-2)\,b_2\frac{\partial}{\partial b_1}+\ldots.+(n-2)\,c_1\frac{\partial}{\partial c_0}+\ldots. \end{aligned}\right\};$$

third, the two which serve to give the development of the concomitant in powers of u and to determine p are

$$\left.\begin{aligned} D_2 &= a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+\ldots+b_0\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial b_2}+\ldots+c_0\frac{\partial}{\partial c_1}+\ldots \\ D_4 &= a_0\frac{\partial}{\partial b_0}+2b_0\frac{\partial}{\partial c_0}+3c_0\frac{\partial}{\partial d_0}+\ldots+a_1\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial c_1}+\ldots+a_2\frac{\partial}{\partial b_2}+\ldots \end{aligned}\right\};$$

and fourth, two of the three operators connected with the subsidiary equations (2) are

$$\left.\begin{aligned} D_7 = 0\,.\,a_0\frac{\partial}{\partial a_0} &+ b_0\frac{\partial}{\partial b_0}+2c_0\frac{\partial}{\partial c_0}+3d_0\frac{\partial}{\partial d_0}+\ldots \\ &+ (-1)\,a_1\frac{\partial}{\partial a_1}+0\,.\,b_1\frac{\partial}{\partial b_1}+1\,.\,c_1\frac{\partial}{\partial c_1}+\ldots \\ &+ (-2)\,a_2\frac{\partial}{\partial a_2}+(-1)\,b_2\frac{\partial}{\partial b_2}+0\,.\,c_2\frac{\partial}{\partial c_2}+\ldots+(-3)\,a_3\frac{\partial}{\partial a_3}+\ldots, \\ D_9 = na_0\frac{\partial}{\partial a_0} &+ (n-2)\,b_0\frac{\partial}{\partial b_0}+(n-4)\,c_0\frac{\partial}{\partial c_0}+\ldots \\ &\qquad + (n-1)\,a_1\frac{\partial}{\partial a_1}+(n-3)\,b_1\frac{\partial}{\partial b_1}+(n-5)\,c_1\frac{\partial}{\partial c_1}+\ldots \\ &\qquad\qquad + (n-2)\,a_2\frac{\partial}{\partial a_2}+(n-4)\,b_2\frac{\partial}{\partial b_2}+\ldots \\ &\qquad\qquad\qquad + (n-3)\,a_3\frac{\partial}{\partial a_3}+\ldots \end{aligned}\right\}$$

17. In obtaining the leading coefficients, which are the simultaneous solutions of the equations $D_1 = 0$ and $D_6 = 0$, the special form of those two equations leads to an easy method of classifying the solutions.

They may be regarded in two ways: first, they are the differential equations of the *simultaneous invariants of the system of binary quantics*

$$\begin{array}{l} (b_0,\ a_1 \between X,\ Y), \\ (c_0,\ b_1,\ a_2 \between X,\ Y)^2, \\ (d_0,\ c_1,\ b_2,\ a_3 \between X,\ Y)^3, \\ (e_0,\ d_1,\ c_2,\ b_3,\ a_4 \between X,\ Y)^4; \\ \dots\dots\dots\dots \end{array}$$

second (which is practically another form of considering the first, and which is the way in which they will be regarded in subsequent applications), they are the differential equations of the *invariants and covariants of a system of simultaneous binary quantics, the coefficients of which are*

$$\begin{array}{l} c_0,\ b_1,\ a_2; \\ d_0,\ c_1,\ b_2,\ a_3; \\ e_0,\ d_1,\ c_2,\ b_3,\ a_4; \\ \dots\dots\dots\dots \end{array}$$

and the variables of which (and of their covariants) are a_1 *and* $-b_0$.

This interpretation is limited to unipartite quantics; a similar interpretation is found subsequently for bipartite quantics. It is to be noticed that, in the present case, the coefficient a_0 of the leading term of the fundamental quantic does not occur in either D_1 or D_6; and it is therefore a simultaneous solution which must, for the aggregate, be associated with the system of simultaneous binariants. With this addition, and regarding a_0 as a binary quantic of order zero, the former of the interpretations of the characteristic equations now is: they are the differential equations of the *simultaneous invariants of the system of binary quantics*

$$\begin{array}{l} (a_0 \between X,\ Y)^0, \\ (b_0,\ a_1 \between X,\ Y)^1, \\ (c_0,\ b_1,\ a_2 \between X,\ Y)^2, \\ (d_0,\ c_1,\ b_2,\ a_3 \between X,\ Y)^3, \\ (e_0,\ d_1,\ c_2,\ b_3,\ a_4 \between X,\ Y)^4, \\ \dots\dots\dots\dots \end{array}$$

18. In the successive applications we shall limit the investigations to the deductions of concomitants which are algebraically independent of one another, and it is therefore necessary to indicate the number of these concomitants which must be expected in any case.

Let N be the total number of literal coefficients which exist in the most general forms of the quantic or of the system of simultaneous quantics, the algebraically independent concomitants of which are desired. Then, when the linear transformations are effected on the quantics, there are N equations connecting the new coefficients with the old; and all these equations involve the elements of the transformation.

In addition to these equations, we have three connecting the variables x and X, and three connecting the variables U and u; also one more given by

$$\Delta = \text{determinant of transformation.}$$

There are thus in all $N + 3 + 3 + 1 = N + 7$ equations which involve the elements of the transformation.

These $N + 7$ are not, however, independent; their number must be reduced by unity on account of the relation

$$u_x = U_X,$$

which, independent of the elements, is satisfied for all linear transformations. The number of equations, independent of one another and involving the elements of the transformation, is therefore

$$N + 6.$$

The number of elements of the transformation is 9.

When, therefore, we proceed to eliminate, among the independent equations, the elements of the transformation, we shall in the end have

$$(N + 6 - 9 =) N - 3$$

relations independent of one another, these relations involving x, X; u, U; Δ, and the coefficients. But we otherwise know that these relations are of the form

$$\phi(A, X, U) = \Delta^\lambda \phi(a, x, u),$$

and every such relation determines a concomitant. Hence *the number of concomitants algebraically independent of one another is* $N - 3$; *and in terms of these every concomitant can be algebraically expressed.*

It is to be understood that, in addition to these, we have the universal concomitant u_x.

A method of obtaining this number is indicated in §35 after some simple cases have been discussed.

PART II.

APPLICATIONS TO UNIPARTITE QUANTICS.

I.—*The Quadratic.*

19. In this case there are six coefficients, and the equation $D_1\psi = 0$ is

$$a_1 \frac{\partial\psi}{\partial b_0} + a_2 \frac{\partial\psi}{\partial b_1} + 2b_1 \frac{\partial\psi}{\partial c_0} = 0.$$

Forming the auxiliary equations necessary to obtain the most general solution of this equation, we have

$$\frac{da_0}{0} = \frac{da_1}{0} = \frac{da_2}{0} = \frac{db_0}{a_1} = \frac{db_1}{a_2} = \frac{dc_0}{2b_1}.$$

Of these five auxiliary equations it is necessary to have five independent integrals, which may evidently be taken in the form

$$\begin{aligned} \theta_0 &= a_0, \\ \theta_1 &= a_1, \\ \theta_2 &= a_2, \\ \theta_3 &= a_2 b_0 - a_1 b_1, \\ \theta_4 &= a_2 c_0 - b_1^2. \end{aligned}$$

Every solution ψ of the equation $D_1\psi = 0$ can, by the theory of this class of differential equations, be expressed as a functional combination of $\theta_0, \theta_1, \theta_2, \theta_3, \theta_4$; and therefore, to obtain solutions common to $D_1\psi = 0$, $D_6\psi = 0$, we must take such functional combinations of $\theta_0, \theta_1, \theta_2, \theta_3, \theta_4$ as satisfy $D_6\psi = 0$. Now

$$\begin{aligned} D_6\theta_0 &= 0, \\ D_6\theta_4 &= 0, \end{aligned}$$

so that θ_0 and θ_4 are common solutions of the two equations. And

$$\begin{aligned} D_6\theta_1 &= b_0, \\ D_6\theta_2 &= 2b_1, \\ D_6\theta_3 &= b_1 b_0 - a_1 c_0, \end{aligned}$$

so that

$$\theta_2 D_6 \theta_1 - \tfrac{1}{2}\theta_1 D_6 \theta_2 = \theta_3,$$

$$\theta_2 D_6 \theta_3 - \tfrac{1}{2}\theta_3 D_6 \theta_2 = -\theta_1 \theta_4.$$

Hence, writing p for $\theta_1 \theta_2^{-\frac{1}{2}}$ and q for $\theta_3 \theta_2^{-\frac{1}{2}}$, we have

$$\theta_2 D_6 p = q, \quad \theta_2 D_6 q = -p\theta_4;$$

and therefore, since $D_6 \theta_4 = 0$, we find

$$D_6 (q^2 + p^2 \theta_4) = 0,$$

so that the only functional combination other than θ_0 and θ_4 is

$$\begin{aligned}\phi &= q^2 + p^2\theta_4\\ &= (\theta_3^2 + \theta_1^2\theta_4)\,\theta_2^{-1}\\ &= a_2 b_0^2 - 2a_1 b_1 b_0 + a_1^2 c_0,\end{aligned}$$

after substitution and reduction. And it follows that every solution common to $D_1\psi = 0$, $D_6\psi = 0$ is expressible in terms of θ_0, θ_4 and ϕ.

The subsidiary operators D_7 and D_9 are, for the present case,

$$D_7 = b_0 \frac{\partial}{\partial b_0} + 2c_0 \frac{\partial}{\partial c_0} - a_1 \frac{\partial}{\partial a_1} - 2a_2 \frac{\partial}{\partial a_2},$$

$$D_9 = 2a_0 \frac{\partial}{\partial a_0} - 2c_0 \frac{\partial}{\partial c_0} + a_1 \frac{\partial}{\partial a_1} - b_1 \frac{\partial}{\partial b_1}.$$

And by actual substitution we find

$$\begin{aligned} D_7\theta_0 &= 0, & D_7\theta_4 &= 0, & D_7\phi &= 0;\\ D_9\theta_0 &= 2\theta_0, & D_9\theta_4 &= -2\theta_4, & D_9\phi &= 0;\end{aligned}$$

so that the subsidiary equations are satisfied provided the values of $m - p$ associated with θ_0, θ_4, and ϕ respectively are $2, -2, 0$. That these are the values may be verified at once by the method of §14.

20. Considering now θ_0, we have $m - p = 2$, so that we determine p. But

$$D_2\theta_0 = \left(a_0 \frac{\partial}{\partial a_1} + b_0 \frac{\partial}{\partial b_1} + 2a_1 \frac{\partial}{\partial a_2}\right)\theta_0 = 0;$$

by §7 it follows that $p = 0$ and therefore $m = 2$. Hence the concomitant is

$$U = a_0 x_1^2 + \ldots,$$

that is, it is the original quantic.

Considering now θ_4, we have $m-p=-2$, so that we determine m. But

$$D_3\theta_4=\left(c_0\frac{\partial}{\partial b_0}+2b_0\frac{\partial}{\partial a_0}+b_1\frac{\partial}{\partial a_1}\right)\theta_4=0\,;$$

so that by §8 it follows that $m=0$ and therefore $p=2$. Hence the concomitant is

$$\Theta=(a_2c_0-b_1^2)\,u_1^2+\ldots\,,$$

that is, it is the reciprocant of the original quantic.

Considering now ϕ, we have $m-p=0$, so that we determine either m or p. We have

$$D_2^2\phi=2\,(a_0^2c_0-a_0b_0^2),$$
$$D_2^3\phi=0,$$

so that by §7 it follows that $p=2$ and therefore $m=2$. Hence the concomitant is

$$\Phi=(a_2b_0^2-2a_1b_1b_0+a_1^2c_0)\,x_1^2u_1^2+\ldots\,.$$

Hence:

Every concomitant of the quadratic can be expressed in terms of U, Θ, and Φ.

21. A simple illustration of the principle of equivalent systems arises in the present case. We have

$$D_2\phi=-2a_0\,(b_1b_0-a_1c_0),\quad D_2\theta_0=0,$$
$$D_2\theta_4=\quad 2\,(a_1c_0-b_0b_1),$$

so that

$$D_2\,(\theta_0\theta_4-\phi)=0.$$

Hence $\theta_0\theta_4-\phi$, as the leading coefficient of a concomitant, has $p=0$; and evidently from the combination of θ_0 and θ_4 it has $m-p=0$, so that $m=0$; the function is an invariant. We therefore take

$$\begin{aligned}H&=\theta_0\theta_4-\phi\\&=a_0c_0a_2+2a_1b_1b_0-a_0b_1^2-a_2b_0^2-a_1^2c_0\,;\end{aligned}$$

and we evidently have

$$u_x^2H=U\Theta-\Phi,$$

so that we have an algebraically equivalent system given by U, Θ, H; and in terms of these three concomitants every concomitant can be expressed. This is the ordinary theory of the quadratic.

It will be noticed that the foregoing method of operators determines a non-resoluble concomitant from a given leading coefficient; the combination $\theta_0\theta_4-\phi$ determines H, and not u_x^2H.

II.—*The Cubic.*

22. In this case there are ten coefficients, and the characteristic equation $D_1\psi = 0$ is

$$a_1\frac{\partial\psi}{\partial b_0} + a_2\frac{\partial\psi}{\partial b_1} + a_3\frac{\partial\psi}{\partial b_2} + 2b_1\frac{\partial\psi}{\partial c_0} + 2b_2\frac{\partial\psi}{\partial c_1} + 3c_1\frac{\partial\psi}{\partial d_0} = 0.$$

Proceeding to obtain the general solution in the usual way, we form the auxiliary equations

$$\frac{da_0}{0} = \frac{da_1}{0} = \frac{da_2}{0} = \frac{da_3}{0} = \frac{db_0}{a_1} = \frac{dc_0}{2b_1} = \frac{dd_0}{3c_1} = \frac{db_1}{a_2} = \frac{dc_1}{2b_2} = \frac{db_2}{a_3},$$

nine in number; it is necessary that we should have nine independent integrals of these auxiliary equations in order to form the most general solution possible of $D_1\psi = 0$.

Now systems of nine independent integrals can be formed in several ways, and these systems are equivalent to one another; that is to say, they are such that the functions occurring in any one system can be uniquely expressed in terms of the functions occurring in any other system. Thus we may take

$$\begin{aligned}
&a_0,\ a_1,\ a_2,\ a_3,\\
&a_2b_0^2 - 2a_1b_1b_0 + a_1^2c_0,\\
&a_3b_0 - a_1b_2,\\
&a_2c_0 - b_1^2,\\
&a_3c_1 - b_2^2,\\
&a_3^2d_0 - 3a_3b_2c_1 + 2b_2^3,
\end{aligned}$$

which are independent of one another, and will therefore constitute a system of the kind required. And modifications can evidently be made among the members of a system, provided that the proper number of independent functions remain; thus, in virtue of the relation

$$a_2(a_2b_0^2 - 2a_1b_1b_0 + a_1^2c_0) = (a_2b_0 - a_1b_1)^2 + a_1^2(a_2c_0 - b_1^2),$$

we may replace a_2 by $a_2b_0 - a_1b_1$.

But it appears on trial (the work is not here reproduced) that the equations similar to those of §19, necessary for the deduction of those functional combinations of the integrals of the foregoing system which will satify $D_6\psi = 0$, are difficult to solve, though not at first sight difficult to form. The system of integrals, which seems to be the easiest to treat in this regard, is obtained by a

method similar to that which is adopted in a corresponding question in the theory of functional invariants* whereby one of the variables of the equation $D_1\psi = 0$ is made, so to speak, a "variable of reference"—a quantity in powers of which integrals are expressed. And the method has the additional advantage of an obvious purely mechanical generalization to quantics of any order.

23. We take, then, as the first of the system of nine integrals

$$\theta_0 = a_0$$

and have at once, replacing D_6 by Δ for convenience,

$$\Delta\theta_0 = 0,$$

so that a_0 is a solution common to the two characteristic equations.

We take as the second of the system

$$\theta_1 = a_1$$

and have

$$\Delta\theta_1 = b_0;$$

this quantity a_1 is taken as the "variable of reference."

We take as the third of the system

$$\theta_2 = a_2,$$

so that $\Delta\theta_2 = 2b_1$, and therefore

$$\theta_1\Delta\theta_2 - 2\theta_2\Delta\theta_1 = 2(a_1b_1 - a_2b_0).$$

Now it may be at once verified that

$$\theta_3 = a_1b_1 - a_2b_0$$

is a solution of the auxiliary equations, and so may be taken as the fourth of the system; and since

$$\Delta\theta_3 = a_1c_0 - b_1b_0,$$

we have

$$\theta_1\Delta\theta_3 - \theta_3\Delta\theta_1 = a_1^2c_0 - 2a_1b_1b_0 + a_2b_0^2.$$

It can similarly be verified that

$$\theta_4 = a_1^2c_0 - 2a_1b_1b_0 + a_2b_0^2$$

is a solution of the auxiliary equations, and it is therefore taken as the fifth of the system. And since

$$\Delta\theta_4 = 0,$$

it is a solution common to the two characteristic equations.

---

* "A class of functional invariants." Phil. Trans. (1889, A), pp. 71–118.

Proceeding similarly, we take as the sixth of the system

$$\theta_5 = a_3,$$

so that, since $\Delta\theta_5 = 3b_2$, we have

$$\theta_1\Delta\theta_5 - 3\theta_5\Delta\theta_1 = 3(a_1b_2 - a_3b_0).$$

As before, it may be verified that

$$\theta_6 = a_1b_2 - a_3b_0$$

is a solution of the auxiliary equations, and may therefore be taken as the seventh of the system; and

$$\Delta\theta_6 = 2(a_1c_1 - b_0b_2),$$

so that
$$\theta_1\Delta\theta_6 - 2\theta_6\Delta\theta_1 = 2(a_1^2c_1 - 2a_1b_0b_2 + a_3b_0^2).$$

Again, the quantity
$$\theta_7 = \quad a_1^2c_1 - 2a_1b_0b_2 + a_3b_0^2$$

is a solution of the auxiliary equations, and it may therefore be taken as the eighth of the system. We have

$$\Delta\theta_7 = a_1^2d_0 - 2a_1b_0c_1 + b_0^2b_2,$$

so that
$$\theta_1\Delta\theta_7 - \theta_7\Delta\theta_1 = a_1^3d_0 - 3a_1^2b_0c_1 + 3a_1b_0^2b_2 - a_3b_0^3.$$

Lastly, the quantity

$$\theta_8 = a_1^3d_0 - 3a_1^2b_0c_1 + 3a_1b_0^2b_2 - a_3b_0^3$$

is a solution of the auxiliary equations, and it may therefore be taken as the ninth of the system. Moreover, since

$$\Delta\theta_8 = 0,$$

it is a solution common to the two equations $D_1\psi = 0$, $D_6\psi = 0$.

It follows from the forms of the nine quantities θ that they are independent of one another, for a_0 is introduced into the system by θ_0, a_1 by θ_1, a_2 by θ_2, b_1 by θ_3, c_0 by θ_4, a_3 by θ_5, b_2 by θ_6, c_1 by θ_7 and d_0 by θ_8 alone, so that among the quantities θ there can be no relation.

24. The process here adopted is one of general application. We take as the first integral of the auxiliary equations the coefficient of the highest power of x_1, and as the "variable of reference," the coefficient of the next lower power of x_1 which involves x_3; starting-points in the succession of integrals are given by coefficients of the terms in the quantic involving x_1 and x_3 only, and successive integrals are suggested by framing combinations of the type $\theta_1\Delta\theta_m - \lambda\theta_m\Delta\theta_1$. These combinations will be called Jacobian combinations.

25. Returning, now, to the equations given by the Jacobian combinations of the quantities θ with θ_1, and modifying them by the substitutions

$$\theta_0 = \phi_0;$$
$$\theta_2 = \theta_1^2\phi_2, \ \theta_3 = \theta_1\phi_3, \ \theta_4 = \phi_4;$$
$$\theta_5 = \theta_1^3\phi_5, \ \theta_6 = \theta_1^2\phi_6, \ \theta_7 = \theta_1\phi_7, \ \theta_8 = \phi_8;$$

we have them in the form

$$\theta_1^2\Delta\phi_0 = 0;$$
$$\theta_1^2\Delta\phi_2 = 2\phi_3,$$
$$\theta_1^2\Delta\phi_3 = \phi_4,$$
$$\theta_1^2\Delta\phi_4 = 0;$$
$$\theta_1^2\Delta\phi_5 = 3\phi_6,$$
$$\theta_1^2\Delta\phi_6 = 2\phi_7,$$
$$\theta_1^2\Delta\phi_7 = \phi_8,$$
$$\theta_1^2\Delta\phi_8 = 0.$$

These are the equations which, when integrated, determine those functional combinations of the quantities θ which are to be solutions of the equation $D_6\psi = 0$. There must (§18) be obtained for this purpose seven independent integrals of these equations, and they may be taken in the forms

$$\chi_1 = \phi_0,$$
$$\chi_2 = \phi_4,$$
$$\chi_3 = \phi_2\phi_4 - \phi_3^2,$$
$$\chi_4 = \phi_8,$$
$$\chi_5 = \phi_6\phi_8 - \phi_7^2,$$
$$\chi_6 = \phi_5\phi_8^2 - 3\phi_6\phi_7\phi_8 + 2\phi_7^3,$$
$$\chi_7 = \phi_3\phi_8 - \phi_4\phi_7.$$

Every solution of the equation $D_6\psi = 0$, which is also a function of the quantities θ or the quantities ϕ and is therefore a solution of the equation $D_1\psi = 0$, can be expressed in terms of the quantities $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7$.

Hence *every solution common to the two equations* $D_1\psi = 0$, $D_6\psi = 0$, *can be expressed in terms of these seven solutions common to the two equations and algebraically independent of one another.*

26. The effects of the operators D_7 and D_9 on the quantities θ are as follows:

	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
D_7	0	$-\theta_1$	$-2\theta_2$	$-\theta_3$	0	$-3\theta_5$	$-2\theta_6$	$-\theta_7$	0
D_9	$3\theta_0$	$2\theta_2$	θ_2	$2\theta_3$	3θ	0	θ_6	$2\theta_7$	$3\theta_8$

and therefore

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
D_7	0	0	0	0	0	0	0
D_9	$3\chi_1$	$3\chi_2$	0	$3\chi_4$	0	0	$3\chi_7$
Value of $m-p$	3	3	0	3	0	0	3

If, then, when the seven quantities χ are expressed in terms of the original coefficients of the quantic, the several values of $m-p$ for each of the quantities derived by the method of §14 should agree with the values as given in the last line of the table; then, by the preceding theory, each of the quantities χ is the leading coefficient, or the source, of a concomitant the full expression of which is obtainable by the methods previously given.

27. When actual substitution is made in the quantities χ and the resulting expressions are reduced, the following are their respective values:

$$\left.\begin{aligned}
v_0 &= \chi_1 = a_0, \\
v_2 &= \chi_2 = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0)^2, \\
h_2 &= \chi_3 = a_2c_0 - b_1^2, \\
v_3 &= \chi_4 = (d_0,\ c_1,\ b_2,\ a_3 \between a_1,\ -b_0)^3, \\
h_3 &= \chi_5 = \boxed{\begin{array}{l|l|l} b_2d_0 & -b_2c_1 & a_3c_1 \\ -c_1^2 & +a_3d_0 & -b_2^2 \end{array}} \between a_1,\ -b_0)^2, \\
\phi_3 &= \chi_6 = \boxed{\begin{array}{l|l|l|l} a_3d_0^2 & a_3c_1d_0 & -a_3b_2d_0 & -a_3^2d_0 \\ -3b_2c_1d_0 & -2b_2^2d_0 & +2a_3c_1^2 & +3a_3b_2c_1 \\ +2c_1^3 & +b_2c_1^2 & -b_2^2c_1 & -2b_2^3 \end{array}} \between a_1,\ -b_0)^3, \\
j_{2,3} &= \chi_7 = \boxed{\begin{array}{l|l|l|l} b_1d_0 & a_2d_0 & -a_3c_0 & -a_3b_1 \\ -c_0c_1 & -2b_2c_0 & +2a_2c_1 & +a_2b_2 \\ & +b_1c_1 & -b_1b_2 & \end{array}} \between a_1,\ -b_0)^3.
\end{aligned}\right\} \quad (13)$$

The determination of the quantities $m-p$ to be associated with these is made by the method of §14. Thus, in the case of χ_4, the weight of any term as $a_1^3 d_0$ is $W = 6\rho - 3$, while that of the term $c_1^3 a_0$, obtained from this by interchanging the coefficients of similar terms in x_1 and x_2, is $W' = 9\rho - 6$, so that for

$$m - p = \frac{W' - W}{\rho - 1} = 3,$$

agreeing with the value in the table. The values of $m-p$ thus derived for the quantities χ agree with the values in the table; and hence we infer that these quantities χ are leading coefficients of concomitants of the cubic. These concomitants will be denoted by U_0; U_2, H_2; U_3, H_3, Φ_3; $J_{2,3}$.

28. We now infer the general theorem:

Every concomitant of the ternary cubic can be expressed as a function of the concomitants U_0; U_2, H_2; U_3, H_3, Φ_3; $J_{2,3}$, *and the universal concomitant* u_x.

The universal concomitant u_x needs to be included; for any constant, say unity, is evidently a solution of the characteristic equations and yet the expression in terms of the seven concomitants is nugatory. The development of the concomitant is normal; for since $\psi_{0,0} = 1$ we have $m = p$ and $\phi_{0,0} = u_1$, so that $m = p = 1$. Also $\phi_{1,0} = u_2$; $\phi_{0,1} = u_3$, and thus we have $x_1 u_1 + x_2 u_2 + x_3 u_3$ as the full expression.

It is a known theorem—here practically proved again—that u_x is the only irreducible universal concomitant for ternary quantics; it is therefore the only one that needs to be associated with the algebraically irreducible concomitants special to any quantic. And it is of importance in the expressions of algebraically reducible concomitants; for when a relation obtained through the leading coefficients is changed into one between the concomitants, it is necessary to associate with each term in the relation such a power of u_x as will render the order and the class uniform throughout.

29. Before these concomitants can be considered as fully given, it is yet necessary to determine for such of them one of the integers m or p and infer the other from the value of $m-p$.

First, for χ_1, we have $m - p = 3$, so that we determine p, using (9). Evidently $D_2 \chi_1 = 0$, so that $p = 0$, and m is therefore 3. Thus

$$U_1 = \chi_1 x_1^3 + \ldots \qquad (14),$$

being in fact the quantic itself.

Second, for χ_2, we have $m - p = 3$, so that we determine p. We have

$$D_2\chi_2 = 2a_0(a_1c_0 - b_1b_0),$$
$$D_2^2\chi_2 = 2a_0(a_0c_0 - b_0^2),$$
$$D_2^3\chi_2 = 0,$$

so that $p = 2$ and m is therefore 5. Thus

$$U_2 = \chi_2 x_1^5 u_1^2 + \ldots . \qquad (15).$$

Third, for χ_3, we have $m - p = 0$. To determine p we have

$$D_2\chi_3 = 2(a_1c_0 - b_1b_0),$$
$$D_2^2\chi_3 = 2(a_0c_0 - b_0^2),$$
$$D_2^3\chi_3 = 0,$$

so that $p = 2 = m$. Thus

$$H_2 = \chi_3 x_1^2 u_1^2 + \ldots . \qquad (16).$$

In this connection one result may be noticed. From the forms of $D_2\chi_2$ and $D_2\chi_3$, we have

$$D_2(\chi_2 - \chi_1\chi_3) = 0.$$

The value of $m - p$ to be associated with $\chi_2 - \chi_1\chi_3$ is 3, and the equation just obtained shows that p is zero, so that $\chi_2 - \chi_1\chi_3$ is the leading coefficient of a pure covariant of the third order. It is the Hessian H, and we have the relation

$$u_x^2 H = U_2 - U_1 H_2.$$

Fourth, for χ_4, for which $m - p = 3$, we determine p. We have

$$D_2^3\chi_4 = 6a_0(a_0^3 d_0 - 3a_0b_0 + 2b_0^3),$$

so that $D_2^4\chi_4 = 0$; hence $p = 3$ and m is therefore 6. Thus

$$U_3 = \chi_4 x_1^6 u_1^3 + \ldots . \qquad (17).$$

Fifth, for χ_5, for which $m = p$. We find in a similar manner that $p = 4$, so that $m = 4$, and thus

$$H_3 = \chi_5 x_1^4 u_1^4 + \ldots . \qquad (18).$$

Similarly, for χ_6, we find $p = 6 = m$, so that

$$\Phi_3 = \chi_6 x_1^6 u_1^6 + \ldots \qquad (19),$$

and for χ_7 we find $p = 4$ and $m = 7$, so that

$$J_{2,3} = \chi_7 x_1^7 u_1^4 + \ldots . \qquad (20).$$

It need hardly be remarked that in order to obtain the complete expressions of these seven fundamental concomitants, it is not necessary to effect all the differentiations implied by (I) and (II) for all the terms; for, since any concomitant is symmetric, either direct or skew, in variables and coefficients of terms of similar rank in the quantic, the knowledge of one term is sufficient to give by symmetric interchange all terms of similar rank in the concomitant.

30. The important connection of §17 between the theory of ternary quantics and that of simultaneous binary quantics, can here be made applicable immediately to the cubic. On inspection of the equations, it appears that they are the characteristic differential equations satisfied by covariants (and invariants) of the simultaneous binary quadratic and binary cubic having a_1 and $-b_0$ for variables; and that therefore *every simultaneous invariant or covariant of the binary quadratic and binary cubic having a_1 and $-b_0$ for variables, is the leading coefficient of a concomitant of the ternary cubic.* Moreover, this is practically the actual form in which the sources have been obtained, and it is on this account that the symbols v_2 (the binary quadratic), h_2 (the Hessian or discriminant of the quadratic), v_3 (the binary cubic), h_3 (the Hessian of the cubic), ϕ_3 (the cubicovariant of the cubic), and J_{23} (the Jacobian of the quadratic and the cubic), have been assigned to the sources.

Hence known results from the theory of the binary concomitants can be used for the theory of the ternary cubic; and conversely, the foregoing theory leads to an important inference as regards simultaneous binary quantics. The quantity a_0, thus interpreted, is in fact the binary quantic of order zero; that is, in the theory of simultaneous binary quantics it is a pure constant, and hence it follows that *every invariant and every covariant in the system which belongs to a binary cubic and a binary quadratic can be algebraically expressed in terms of the quadratic and its discriminant, the cubic with its Hessian and cubicovariant, and the Jacobian of the cubic and the quadratic.**

31. As illustrations of the general theorem that all the concomitants of the ternary quantic are expressible in terms of the given system, we proceed to some special cases.

---

*For the theory of the asyzygetic concomitants of the binary quadratic and binary cubic when simultaneous, see Salmon's *Higher Algebra*, §149, in 3d edition (1876); Clebsch's *Theorie der binären algebraischen Formen*, §59; Gordan's *Vorlesungen über Invariantentheorie*, §31 (a comparison of their notations is given in my memoir, "A class of functional invariants," Phil. Trans., 1889, A, p. 91, note), and Hammond, *The cubi-quadric system*, Amer. Journ. of Math., Vol. VIII (1886), pp. 138–155.

The cubic contravariant P has for its leading coefficient*

$$p = c_0 (a_3 c_1 - b_2^2) - b_1 (a_3 d_0 - b_2 c_1) + a_2 (b_2 d_0 - c_1^2),$$

which is the invariant I of Salmon, the intermediate invariant of v_2 and h_3. Now if K be the Jacobian of two quadratics θ_1 and θ_2 of which the discriminants are D_1 and D_2 and intermediate invariant is I, then

$$K^2 = \theta_1 \theta_2 I - \theta_1^2 D_2 - \theta_2^2 D_1.$$

Taking θ_2 to be v_2, we have $D_2 = h_2$; and taking θ_1 to be h_3, we have

$$\begin{aligned} 4D_1 &= 4 (b_2 d_0 - c_1^2)(a_3 c_1 - b_2^2) - (a_3 d_0 - b_2 c_1)^2 \\ &= - \text{discriminant of cubic} \\ &= - \frac{1}{v_3^2} (\phi_3^2 + 4h_3^3). \end{aligned}$$

Also

$$K v_3 = j_{23} h_3 + \frac{1}{2} v_2 \phi_3,$$

for K, j_{23}, ϕ_3 are all Jacobians. Hence the relation becomes

$$\begin{aligned} p v_2 h_3 v_3^2 &= \left(j_{23} h_3 + \frac{1}{2} v_2 \phi_3\right)^2 - \frac{1}{4} v_2^2 (\phi_3^2 + 4h_3^3) + v_3^2 h_3^2 h_2 \\ &= j_{23}^2 h_3^2 + j_{23} v_2 \phi_3 h_3 - v_2^2 h_3^3 + v_3^2 h_3^2 h_2, \end{aligned}$$

so that

$$p v_2 v_3^2 = j_{23}^2 h_3 + j_{23} v_2 \phi_3 - v_2^2 h_3^2 + v_3^2 h_2 h_3.$$

Hence passing to the concomitants of which the foregoing quantities are leading coefficients, we have

$$P u_x U_2 U_3^2 = J_{23}^2 H_3 + J_{23} U_2 \Phi_3 - U_2^2 H_3^2 + U_3^2 H_2 H_3,$$

the factor u_x being inserted to make the order and the class uniform.

The reciprocant F (Cayley, l. c., p. 644) has for its leading coefficient

$$\begin{aligned} f &= a_3^2 d_0^2 - 6 a_3 b_2 c_1 d_0 + 4 a_3 c_1^3 + 4 d_0 b_2^3 - 3 b_2^2 c_1^2 \\ &= \frac{1}{v_3^2} (\phi_3^2 + 4h_3^3), \end{aligned}$$

and therefore

$$F U_3^2 = \Phi_3^2 + 4 H_3^3,$$

no factor u_x being necessary.

The value of the quartinvariant S (Cayley, l. c., p. 641) is

$$S = v_0 p - h_3 + h_2^2 + l_2$$

---

* Cayley, *Third Memoir on Quantics*, Phil. Trans., 1856, p. 642.

(in the table the quantity $-cf^2h$ should be $-cfh^2$), where

$$l_2 = a_1\{d_0a_2b_1 - c_1(a_2c_0 + 2b_1^2) + 3b_2c_0b_1 - a_3c_0^2\} \\ - b_0\{d_0a_2^2 - 3c_1a_2b_1 + b_2(a_2c_0 + 2b_1^2) - a_3c_0b_1\},$$

being in fact the linear covariant L_2 of Salmon ($=q$ of Gordan); and it has already (§30) been proved that l_2 must be expressible in terms of the seven functions (13). In fact it is not difficult to verify that

$$l_2v_2v_3^2 = -\phi_3v_2^3 + j_{23}(h_2v_3^2 + 3h_3v_2^2 + j_{23}^2).$$

Hence

$$Sv_2v_3^2 = v_0v_2v_3^2p - v_2v_3^2h_3 + v_2v_3^2h_2^2 - \phi_3v_2^3 + j_{23}h_2v_3^2 + 3h_3v_2^2j_{23} + j_{23}^3,$$

the relation among leading coefficients. This, when turned into the corresponding relation between the concomitants, is

$$SU_2U_3^2u_x^4 = U_1(J_{23}^2H_3 + J_{23}U_2\Phi_3 - U_2^2H_3^2 + U_3^2H_2H_3) \\ - U_2U_3^2H_3 + U_2U_3^2H_2^2 - \Phi_3U_2^3 + J_{23}H_2U_3^2 + 3H_3U_2^2J_{23} + J_{23}^3,$$

the power of u_x being inserted to make the order and the class uniform throughout the equation.

And in every case the first step in the expression of a given concomitant in terms of the fundamental concomitants is the arrangement of its leading coefficients in powers of a_1 and b_0 (i. e. of h and j in Cayley's tables).

32. As a last illustration we may take the following: It is a consequence of the general theory that each of the eight quantities U, H, Ψ; P, Q, F; S, T* is expressible in terms of the seven fundamental concomitants and of u_x. Hence some relation must subsist among these eight quantities and u_x, an irrational form of which relation can be obtained by the following indications.

Taking U in the form $x^3 + y^3 + z^3 + 6lxyz$, we have

$$\begin{aligned} x^3 + y^3 + z^3 &= \lambda, \\ y^3z^3 + z^3x^3 + x^3y^3 &= \mu, \\ xyz &= r, \end{aligned}$$

where λ, μ, r are expressible in terms of U, H, Ψ and the coefficient l. Now if $x + y + z = 3p$, $yz + zx + xy = 3q$, we have

$$p(p^2 - q) = \tfrac{1}{27}(\lambda - 3r) = \sigma,$$

$$q(q^2 - pr) = \tfrac{1}{27}(3r^2 - \mu) = \rho, \text{ say,}$$

* Cayley's 34 Concomitants of the Ternary Cubic, *Amer. Journ. of Math.*, Vol. IV (1881), p. 1.

where σ and ρ are similarly expressed to λ, μ, r. From these it follows that if $p^3 = Y$, we have

$$Y^3 - Y^2(3\sigma + r) + Y(3\sigma^2 + r\sigma - \rho) - \sigma^3 = 0. \qquad \text{(i)}$$

Similarly, if

$$\begin{aligned} \xi^3 + \eta^3 + \zeta^3 &= \lambda', \\ \eta^3\zeta^3 + \zeta^3\xi^3 + \xi^3\eta^3 &= \mu', \\ \xi\eta\zeta &= r', \end{aligned}$$

where λ', μ', r' are expressible in terms of P, Q, F and the coefficient l, and if we write $\xi + \eta + \zeta = 3p' = 3\sqrt[3]{Y'}$, then

$$Y'^3 - Y'^2(3\sigma' + r') + Y'(3\sigma'^2 + r'\sigma' - \rho') - \sigma'^3 = 0, \qquad \text{(ii)}$$

and both Y and Y' (and therefore p and p') can be expressed in irrational forms by means of (i) and (ii) in terms of the three covariants, the three contravariants and the coefficient l.

Now x, y, z being the roots of

$$u^3 - 3pu^2 + 3qu - r = 0,$$

and ξ, η, ζ those of

$$u^3 - 3p'u^2 + 3q'u - r' = 0,$$

then (Burnside and Panton, Theory of Equations, 1st edit. p. 113)

$$u_x = x\xi + y\eta + z\zeta = 3t$$

satisfies the equation

$$(pp' - t)^3 - 3hh'(pp' - t) + \tfrac{1}{2}(gg' \pm \sqrt{\Delta\Delta'}) = 0, \qquad \text{(iii)}$$

where

$$\begin{aligned} h &= q - p^2 = -\frac{\sigma}{p}, \\ g &= -r + 3pq - 2p^3 = -r - 3\sigma + p^3, \end{aligned}$$

and

$$\Delta = g^2 + 4h^3,$$

with similar values for h', g', Δ'.

When the value of $p\,(=\sqrt[3]{Y})$ derived from (i) is substituted in h, g, Δ, they are expressed (irrationally) in terms of l and the three covariants; and when the value of $p'\,(=\sqrt[3]{Y'})$ derived from (ii) is substituted in h', g', Δ', they are similarly expressed in terms of l and the three contravariants. When both these sets of quantities and the values of p and p' are substituted in (iii), it comes to be an equation between U, H, Ψ; P, Q, F; u_x, and l. When its rationalized equivalent is obtained, it follows—from the fact that all the occurring quanti-

ties, other than combinations of l, are covariantive—that such combinations are also covariantive; that is to say, they can be expressed in terms of S and T.* This rational form would be the required relation.

Symbolical Representation of the Concomitants.

33. Instead of determining the order and the class by means of equations (9) and (11), the following method is effective, viz. *to change the leading coefficient into one which is symbolical in the umbral elements of the original quantic, and complete this symbolical expression according to the laws which apply to the concomitants of ternary quantics.*

For this purpose let

$$U_1 = a_0 x_1^3 + \ldots\ldots = \alpha_x^3 = \beta_x^3 = \gamma_x^3 = \ldots\ldots$$

Then

$$h_2 = a_2 c_0 - b_1^2 = \tfrac{1}{2}\,\alpha_1\beta_1(\beta_2\alpha_3)^2,$$

and therefore

$$H_2 = \tfrac{1}{2}\,(\alpha\beta u)^2\alpha_x\beta_x.$$

Next we have

$$v_2 = (c_0,\, b_1,\, a_2 \between a_1,\, -b_0)^2 = \alpha_1\varepsilon_\xi^2$$

for $c_0 = \alpha_1\alpha_2^2$, $b_1 = \alpha_1\alpha_2\alpha_3$, $a_2 = \alpha_1\alpha_3^2$, and

$$\begin{aligned}\varepsilon_\xi &= \alpha_2 a_1 - \alpha_3 b_0\\ &= \beta_1^2(\alpha_2\beta_3).\end{aligned}$$

Hence, remembering that with the symbolical notation, the repetition of a real coefficient requires the introduction of a new umbral coefficient, we have

$$v_2 = \alpha_1\cdot\beta_1^2(\alpha_2\beta_3)\cdot\gamma_1^2(\alpha_2\gamma_3),$$

and therefore

$$U_2 = (\alpha\beta u)(\alpha\gamma u)\,\alpha_x\beta_x^2\gamma_x^2.$$

Next we have

$$v_3 = (d_0,\, c_1,\, b_2,\, a_3 \between a_1,\, -b_0)^3 = \theta_\xi^3,$$

where

$$\theta_\xi = \alpha_2 a_1 - \alpha_3 b_0 = (\alpha_2\beta_3)\,\beta_1^2,$$

so that as before

$$v_3 = (\alpha_2\beta_3)\,\beta_1^2\cdot(\alpha_2\gamma_3)\,\gamma_1^2\cdot(\alpha_3\delta_3)\,\delta_1^2,$$

and therefore

$$U_3 = (\alpha\beta u)(\alpha\gamma u)(\alpha\delta u)\,\beta_x^2\gamma_x^2\delta_x^2.$$

---

* It is not sufficient to have one only of the two invariants, for in a less special form we should have the quantities l^3 and abc, in the notation of Cayley's paper on the 34 concomitants.

Further, h_3 is the Hessian of v_3, so that we may write

$$h_3 = \frac{1}{2}\,(\theta\phi)^2)\,\theta_\xi\phi_\xi$$
$$= \frac{1}{2}\,(\alpha_2\beta_3)^2.(\alpha_2\gamma_3)\,\gamma_1^2.(\beta_2\delta_3)\,\delta_1^2,$$

and therefore

$$H_3 = \frac{1}{2}\,(\alpha\beta u)^2(\alpha\gamma u)(\beta\delta u)\,\gamma_x^2\delta_x^2.$$

Again, ϕ_3 is the cubicovariant of v_3, so that

$$\phi_3 = (\theta\phi)^2(\theta\psi)\,\phi_\xi\psi_\xi^2$$
$$= (\alpha_2\beta_3)^2(\alpha_2\gamma_3).(\beta_2\delta_3)\,\delta_1^2\,[(\gamma_2\varepsilon_3)\,\varepsilon_1^2.(\gamma_2\lambda_3)\,\lambda_1^2],$$

and therefore

$$\frac{\Phi}{3} = (\alpha\beta u)^2(\alpha\gamma u)(\beta\delta u)(\gamma\varepsilon u)(\gamma\lambda u)\,\delta_x^2\varepsilon_x^2\lambda_x^2.$$

And lastly, j_{23} is the Jacobian of v_2 and v_3, so that

$$j_{23} = \alpha_1\,(\varepsilon\theta)\,\varepsilon_\xi\theta_\xi^2.$$

We write $\varepsilon_1 = \alpha_2$ and $\varepsilon_2 = \alpha_3$, $\theta_1 = \beta_2$ and $\theta_2 = \beta_3$, and so on, so that

$$j_{23} = \alpha_1\,(\alpha_2\beta_3).(\alpha_2\gamma_3)\,\gamma_1^2\,[(\beta_2\delta_3)\,\delta_x^2.(\beta_2\varepsilon_3)\,\varepsilon_x^2],$$

and therefore

$$J_{23} = (\alpha\beta u)(\alpha\gamma u)(\beta\delta u)(\beta\varepsilon u)\,\alpha_x\gamma_x^2\delta_x^2\varepsilon_x^2.$$

It will be noticed that for each of the functions thus represented, the order and the class agree with their former values.

Modification of the Fundamental System.

34. According to a well known proposition, the square of any Jacobian binariant can be expressed in terms of other concomitants associated with the binary quantics of which the Jacobian is taken, and as the Jacobian may thus be considered to be of ambiguous sign, it may be deemed desirable to replace the two Jacobians in the foregoing system, viz. ϕ_3 and j_{23}, by equivalent concomitants of determinate sign.

For the first of them we have

$$\phi_3^2 = fv_3^2 - 4h_3^3,$$

where f is the discriminant of v_3, already (§31) considered, and thus replace ϕ_3 by f, which written symbolically is

$$(\alpha_2\beta_3)^2(\gamma_2\delta_3)^2(\alpha_2\gamma_3)(\beta_2\delta_3),$$

we have the corresponding function given by

$$F = (\alpha\beta u)^2(\gamma\delta u)^2(\alpha\gamma u)(\beta\delta u).$$

This replaces Φ_3.

To replace j_{23} we take the quantity p (of §31), the symbolical form of which is

$$p = \alpha_1 (\epsilon\eta)^2,$$

where

$$\eta_x^2 = h_3 = \frac{1}{2} (\theta\phi)^2 \theta_x \phi_x,$$

so that

$$p = \frac{1}{2} \alpha_1 (\theta\phi)^2 (\epsilon\theta)(\epsilon\phi)$$

$$= \frac{1}{2} \alpha_1 (\beta_2\gamma_3)^2(\alpha_2\gamma_3)(\alpha_2\beta_3).$$

This leading coefficient determines a function

$$\frac{1}{2} (\beta\gamma u)^2(\alpha\gamma u)(\alpha\beta u)\,\alpha_x$$

$$= \frac{1}{6} (\alpha\beta u)(\alpha\gamma u)(\beta\gamma u)(\alpha\beta\gamma)\,u_x,$$

by the usual method of compounding these determinants; hence there is effectively determined a function

$$P = \frac{1}{6} (\alpha\beta\gamma)(\alpha\beta u)(\alpha\gamma u)(\beta\gamma u).$$

This replaces J_{23}; and the present rejection of the factor u_x accounts for its insertion in §31, necessary to render the order and the class both uniform throughout the equation which gives the expression for P in terms of the former fundamental system.

The Number of Algebraically Independent Concomitants of the Ternary n^{tic}.

35. Before proceeding to the detailed consideration of the quartic, the general method of obtaining the proper (§18) number of the independent solutions of the equations $D_1\psi = 0$, $D_6\psi = 0$ can now be indicated.

We find as before a complete set of independent solutions $\theta_0, \theta_1, \theta_2, \ldots$ of $D_1\psi = 0$, and then take such functional combinations of them, say $f(\theta_0, \theta_1, \ldots)$, as will satisfy $D_6\psi = 0$ or $\Delta\psi = 0$; we must therefore have

$$0 = \frac{\partial f}{\partial\theta_0}\Delta\theta_0 + \frac{\partial f}{\partial\theta_1}\Delta\theta_1 + \frac{\partial f}{\partial\theta_2}\Delta\theta_2 + \ldots.$$

Hence the subsidiary equations necessary for the determination of f are

$$\frac{d\theta_0}{\Delta\theta_0} = \frac{d\theta_1}{\Delta\theta_1} = \frac{d\theta_2}{\Delta\theta_2} = \ldots\ldots,$$

the number of independent solutions of this set will give the required number of independent functional combinations. Now all the quantities $\Delta\theta$ are not functions of the variables θ of these equations; it is necessary to take such combinations of the $\Delta\theta$ as are expressible in terms of the variables. In actual practice these combinations are similar to those which arose in the case of the cubic (§25), viz. functions of quotients of the variables θ such that when a quotient is operated on by Δ, the result is expressible in terms of some other quotient.

To estimate the effect of these modifications, let us consider them in connection with the ternary quantic of order n, which has $\frac{1}{2}(n+1)(n+2)$ coefficients. The number of subsidiary equations associated with $D_1\psi = 0$ is less than this integer by unity, and therefore the number of the quantities θ being the number of independent integrals of these equations, is

$$\frac{1}{2}(n^2 + 3n).$$

In forming the functional combinations of the quantities $\Delta\theta$, it is necessary (§23) to take some one of the quantities θ, as θ_1, for a variable of reference, and then the number of independent equations of the form

$$\theta_1\Delta\theta_r - \lambda\theta_r\Delta\theta_1 = \theta_s,$$

which can be formed, is $\frac{1}{2}(n^2+3n) - 1$. Each such equation can be used for the transformation of a fraction in the subsidiary equations

$$\frac{d\theta_0}{\Delta\theta_0} = \frac{d\theta_1}{\Delta\theta_1} = \ldots\ldots,$$

and therefore the number of equations in the modified set being one less than the number of modified fractions, is $\frac{1}{2}(n^2+3n) - 2$. But each of these modified subsidiary equations leads to an integral, and therefore the number of independent integrals is $\frac{1}{2}(n^2+3n) - 2$, which is the number $\frac{1}{2}(n+1)(n+2) - 3$ of §18,

$$= \frac{1}{2}(n+4)(n-1),$$

which is the required number of algebraically independent solutions of the simultaneous partial differential equations $D_1\psi = 0$, $D_6\psi = 0$.

Since each such solution determines a concomitant, we have the result:

All the concomitants of the uni-ternary n^{tic} can be algebraically expressed in terms of u_x and of $\frac{1}{2}(n+4)(n-1)$ properly chosen independent concomitants.

Thus there are 3 for the quadratic in this algebraically complete system; there are 7 for the cubic, as was proved, and, as we shall now see, there are 12 for the quartic.

In the same way it may be proved that:

All the concomitants of the bi-ternary $n^o m^{ic}$, symbolically represented by $a_x^n u_\alpha^m$, can be algebraically expressed in terms of $\frac{1}{4}(n+1)(n+2)(m+1)(m+2)-3$ properly chosen independent concomitants.

III.—*The Quartic.*

36. The explicit form of the general quartic is

$$\begin{aligned} &a_0x_1^4 + 4x_3a_1x_1^3 + 6x_3^2a_2x_1^2 + 4x_3^3a_3x_1 + a_4x_3^4, \\ &+ 4b_0x_1^3x_2 + 4x_3 3b_1x_1^2x_2 + 6x_3^2 2b_2x_1x_2 + 4x_3^3b_3x_2 \\ &+ 6c_0x_1^2x_2^2 + 4x_3 3c_1x_1x_2^2 + 6x_3^2c_2x_2^2 \\ &+ 4d_0x_1x_2^3 + 4x_3d_1x_2^3 \\ &+ e_0x_2^4, \end{aligned}$$

and the characteristic equations $D_1\psi = 0$, $D_6\psi = 0$ are respectively

$$\begin{aligned} D_1 = a_1\frac{\partial}{\partial b_0} + 2b_1\frac{\partial}{\partial c_0} + 3c_1\frac{\partial}{\partial d_0} + 4d_1\frac{\partial}{\partial e_0} + a_2\frac{\partial}{\partial b_1} + 2b_2\frac{\partial}{\partial c_1} \\ + 3c_2\frac{\partial}{\partial d_1} + a_3\frac{\partial}{\partial b_2} + 2b_3\frac{\partial}{\partial c_2} + a_4\frac{\partial}{\partial b_3} = 0, \end{aligned}$$

$$\begin{aligned} D_6 = \Delta = b_0\frac{\partial}{\partial a_1} + 2b_1\frac{\partial}{\partial a_2} + 3b_2\frac{\partial}{\partial a_3} + 4b_3\frac{\partial}{\partial a_4} + c_0\frac{\partial}{\partial b_1} + 2c_1\frac{\partial}{\partial b_2} \\ + 3c_2\frac{\partial}{\partial b_3} + d_0\frac{\partial}{\partial c_1} + 2d_1\frac{\partial}{\partial c_2} + e_0\frac{\partial}{\partial d_1} = 0. \end{aligned}$$

To find the common solutions, it is first necessary to construct the subsidiary equations for the former of these, being 14 in number since there are 15 coefficients, and so 14 independent integrals of them are necessary.

The form of nine of these subsidiary equations is exactly the same as for the cubic, so that their integrals are the same as for the cubic, viz. we take

$$\theta_0,\ \theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6,\ \theta_7,\ \theta_8,$$

as in §23. For the remainder, which are

$$\frac{da_0}{0} = \ldots = \frac{da_4}{0} = \frac{db_3}{a_4} = \frac{dc_2}{2b_3} = \frac{dd_1}{3c_2} = \frac{de_0}{4d_1},$$

we proceed as before, and take

$$\theta_9 = a_4.$$

Then $\Delta\theta_9 = 4b_3$, so that

$$\theta_1\Delta\theta_9 - 4\theta_9\Delta\theta_1 = 4\theta_{10},$$

where

$$\theta_{10} = b_3a_1 - a_4b_0$$

is the next integral. We now have

$$\theta_1\Delta\theta_{10} - 3\theta_{10}\Delta\theta_1 = 3\theta_{11},$$

where

$$\theta_{11} = c_2a_1^2 - 2b_3a_1b_0 + a_4b_0^2$$

is another integral. Next

$$\theta_1\Delta\theta_{11} - 2\theta_{11}\Delta\theta_1 = 2\theta_{12},$$

where

$$\theta_{12} = d_1a_1^3 - 3c_2a_1^2b_0 + 3b_3a_1b_0^2 - a_4b_0^3$$

is the succeeding integral; and

$$\theta_1\Delta\theta_{12} - \theta_{12}\Delta\theta_1 = \theta_{13},$$

where

$$\theta_{13} = e_0a_1^4 - 4d_1a_1^3b_0 + 6c_2a_1^2b_0^2 - 4b_3a_1b_0^3 + a_4b_0^4$$

is the last integral; moreover, we have

$$\Delta\theta_{13} = 0.$$

37. By the substitutions

$$\theta_9\theta_1^{-4} = \phi_9,\quad \theta_{10}\theta_1^{-3} = \phi_{10},\quad \theta_{11}\theta_1^{-2} = \phi_{11},\quad \theta_{12}\theta_1^{-1} = \phi_{12},\quad \theta_{13}\theta_1^{-0} = \phi_{13},$$

these equations take the same forms as the eight equations of §25, viz.

$$\begin{aligned}\theta_1^2\Delta\phi_9 &= 4\phi_{10},\\ \theta_1^2\Delta\phi_{10} &= 3\phi_{11},\\ \theta_1^2\Delta\phi_{11} &= 2\phi_{12},\\ \theta_1^2\Delta\phi_{12} &= \phi_{13},\\ \theta_1^2\Delta\phi_{13} &= 0\ ,\end{aligned}$$

so that of the $(8+5=)$ 13 equations we must have 12 integrals. Seven of these are already obtained, being $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7$; the remaining five are easily found to be

$$\begin{aligned}
\chi_8 &= \phi_{13}, \\
\chi_9 &= \phi_{11}\phi_{13} - \phi_{12}^2, \\
\chi_{10} &= \phi_{10}\phi_{13}^2 - 3\phi_{11}\phi_{12}\phi_{13} + 2\phi_{12}^3, \\
\chi_{11} &= \phi_9\phi_{13} - 4\phi_{10}\phi_{12} + 3\phi_{11}^2, \\
\chi_{12} &= \phi_3\phi_{13} - \phi_4\phi_{12}.
\end{aligned}$$

The twelve integrals are independent of one another, and every solution common to the two equations $D_1\psi = 0 = D_6\psi$ can be algebraically expressed in terms of $\chi_1, \chi_2, \ldots, \chi_{11}, \chi_{12}$.

38. The effects of the operators D_7 and D_9 on the quantities θ in the case of the quartic are as follows:

	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}
D_7	0	$-\theta_1$	$-2\theta_2$	$-\theta_3$	0	$-3\theta_5$	$-2\theta_6$	$-\theta_7$	0	$-4\theta_9$	$-3\theta_{10}$	$-2\theta_{11}$	$-\theta_{12}$	0
D_9	$4\theta_0$	$3\theta_1$	$2\theta_2$	$4\theta_3$	$6\theta_4$	θ_5	$3\theta_6$	$5\theta_7$	$7\theta_8$	0	$2\theta_{10}$	$4\theta_{11}$	$6\theta_{12}$	$8\theta_{13}$

and therefore

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
D_7	0	0	0	0	0	0	0	0	0	0	0	0
D_9	$4\chi_1$	$6\chi_2$	$2\chi_3$	$7\chi_4$	$4\chi_5$	$6\chi_6$	$8\chi_7$	$8\chi_8$	$6\chi_9$	$9\chi_{10}$	$-4\chi_{11}$	$9\chi_{12}$
Value of $m-p$	4	6	2	7	4	6	8	8	6	9	-4	9

Hence, if ultimately it should appear that the values of $m-p$ to be associated with the respective sources are as given in the last line of the second table, the twelve quantities χ are sources of concomitants.

39. When actual substitution is made in the quantities χ and the resulting expressions are reduced, the values of $\chi_1, \ldots\ldots, \chi_7$ are as given in (13), but it must be remembered that the coefficients are now coefficients of the quartic, and therefore different values of m and of p may need to be associated with those quantities. For the remainder we have

$$v_4 = \chi_8 = (e_0, d_1, c_2, b_3, a_4 \between a_1, -b_0)^4,$$

$$h_4 = \chi_9 = \left|\begin{array}{c|c|c|c|c} +e_0c_2 & +2e_0b_3 & +e_0a_4 & +2d_1a_4 & +a_4c_2 \\ -d_1^2 & -2c_2d_1 & +2d_1b_3 & -2b_3c_2 & -b_3^2 \\ & & -3c_2^2 & & \end{array}\right| \between a_1, -b_0)^4$$

$$\phi_4 = \chi_{10} = \left|\begin{array}{c|c|c|c|c|c|c} +e_0^2b_3 & +e_0^2a_4 & +5e_0d_1a_4 & -10e_0b_3^2 & -5e_0b_3a_4 & -e_0a_4^2 & -d_1^4a_4^2 \\ -3e_0d_1c_2 & +2e_0b_3d_1 & -15e_0c_2b_3 & +10d_1^2a_4 & +15d_1c_2a_4 & -2d_1b_3a_4 & +3c_2b_3a_4 \\ +2d_1^3 & -9e_0c_2^2 & +10d_1^2b_3 & & -10d_1b_3^2 & +9c_2^2a_4 & -2b_3^3 \\ & +6d_1^2c_2 & & & & -6c_2b_3^2 & \end{array}\right| \between a_1, -b_0)^6 \qquad (21)$$

$$i_4 = \chi_{11} = e_0a_4 - 4d_1b_3 + 3c_2^2,$$

$$j_{24} = \chi_{12} = \left|\begin{array}{c|c|c|c|c} +e_0b_1 & +e_0a_2 & +3d_1a_2 & -a_4c_0 & +b_3a_2 \\ -c_0d_1 & -3c_2c_0 & -3b_3c_0 & +3c_2a_2 & -a_4b_1 \\ & +2b_1d_1 & & -2b_3b_1 & \end{array}\right| \between a_1, -b_0)^4$$

The reasons of the notations adopted are fairly obvious; h_4 is the Hessian, ϕ_4 the cubicovariant, i_4 the quadrinvariant, of v_4 regarded as a quartic; and j_{24} is the Jacobian of v_2, regarded as a quadratic, and v_4.

40. To find the values of m and p, the shortest method will be to change all the expressions into symbolical forms. For this purpose, let

$$U_1 = a_0x_1^4 + \ldots\ldots = \alpha_x^4 = \beta_x^4 = \gamma_x^4 = \ldots\ldots$$

be the original quartic. It is evident that U_1 is the concomitant with the source v_0.

Then

$$h_2 = a_2c_0 - b_1^2 = \tfrac{1}{2}\, \alpha_1^2\beta_1^2\,(\beta_2\alpha_3)^2,$$

and therefore

$$H_2 = \tfrac{1}{2}\,(\alpha\beta u)^2\,\alpha_x^2\beta_x^2$$
$$= \chi_3x_1^4u_1^2 + \ldots\ldots \qquad (22).$$

Next we have $$v_2 = (c_0,\ b_1,\ a_2 \between a_1,\ -b_0)^2 = \alpha_1^2 \varepsilon_\xi^2,$$

for $c_0 = \alpha_1^2\alpha_2^2$, $b_1 = \alpha_1^2\alpha_2\alpha_3$, $a_2 = \alpha_1^2\alpha_3^2$; and

$$\begin{aligned} \varepsilon_\xi &= \alpha_2 a_1 - \alpha_3 b_0 \\ &= \beta_1^3 (\alpha_2 \beta_3), \end{aligned}$$

so that $$v_2 = \alpha_1^2 . \beta_1^3 (\alpha_2\beta_3) . \gamma_1^3 (\alpha_2\gamma_3),$$

and therefore $$\begin{aligned} U_2 &= (\alpha\beta u)(\alpha\gamma u)\, \alpha_x^2 \beta_x^3 \gamma_x^3 \\ &= \chi_2 x_1^8 u_1^2 + \dots . \end{aligned} \tag{23}$$

(As in the case of the cubic, we have $U_2 - U_1 H_2$ divisible by u_x^2 and leaving as its other factor the Hessian of the quartic.)

Next we have $$v_3 = (d_0,\ c_1,\ b_2,\ a_3 \between a_1,\ -b_0)^3 = \alpha_1 \theta_\xi^3,$$

where $$\theta_\xi = \alpha_2 a_1 - \alpha_3 b_0 = \beta_1^3 (\alpha_2\beta_3),$$

so that $$v_3 = \alpha_1 . \beta_1^3 (\alpha_2\beta_3) . \gamma_1^3 (\alpha_2\gamma_3) . \delta_1^3 (\alpha_2\delta_3);$$

and therefore $$\begin{aligned} U_3 &= (\alpha\beta u)(\alpha\gamma u)(\alpha\delta u)\, \alpha_x \beta_x^3 \gamma_x^3 \delta_x^3 \\ &= \chi_4 x_1^{10} u_1^3 + \dots . \end{aligned} \tag{24}$$

Next, h_3 is the Hessian of v_3, so that

$$\begin{aligned} h_3 &= \tfrac{1}{2}\, \alpha_1 \beta_1 (\theta\phi)^2 \theta_\xi \phi_\xi \\ &= \tfrac{1}{2}\, \alpha_1 \beta_1 (\alpha_2\beta_3)^2 . (\alpha_2\gamma_3)\, \gamma_1^3 . (\beta_2\delta_3)\, \delta_1^3, \end{aligned}$$

and therefore $$\begin{aligned} H_3 &= \tfrac{1}{2} (\alpha\beta u)^2 (\alpha\gamma u)(\beta\delta u)\, \alpha_x \beta_x \gamma_x^3 \delta_x^3 \\ &= \chi_5 x_1^8 u_1^4 + \dots . \end{aligned} \tag{25}$$

Again, ϕ_3 is the cubicovariant of v_3, so that

$$\begin{aligned} \phi_3 &= \alpha_1 \beta_1 \gamma_1 (\theta\phi)^2 \phi_\xi \psi_\xi^2 (\theta\psi) \\ &= \alpha_1 \beta_1 \gamma_1 (\alpha_2\beta_3)^2 (\alpha_2\gamma_3) . \delta_1^3 (\beta_2\delta_3) . (\gamma_2\lambda_3)\, \lambda_1^3 (\gamma_2\mu_3)\, \mu_1^3, \end{aligned}$$

and therefore $$\begin{aligned} \Phi_3 &= (\alpha\beta u)^2 (\alpha\gamma u)(\beta\delta u)(\gamma\lambda u)(\gamma\mu u)\, \alpha_x \beta_x \gamma_x \delta_x^3 \lambda_x^3 \mu_x^3 \\ &= \chi_6 x_1^{12} u_1^6 + \dots . \end{aligned} \tag{26}$$

Also, j_{23} is the Jacobian of v_2 and v_3, so that

$$\begin{aligned} j_{23} &= \alpha_1^2 \beta_1 (\varepsilon\theta)\, \varepsilon_\xi \theta_\xi^2 \\ &= \alpha_1^2 \beta_1 (\alpha_2\beta_3) . (\alpha_2\gamma_3)\, \gamma_1^3 \left[(\beta_2\delta_3)\, \delta_1^3 . (\beta_2\lambda_3)\, \lambda_1^3 \right], \end{aligned}$$

and therefore

$$J_{23} = (\alpha\beta u)(\alpha\gamma u)(\beta\delta u)(\beta\lambda u)\,\alpha_x^2\beta_x\gamma_x^3\delta_x^3\lambda_x^3$$
$$= \chi_7 x_1^{12}u_1^4 + \dots \qquad (27).$$

Coming now to the new forms in (21), we have

$$v_4 = (e_0,\, d_1,\, c_2,\, b_3,\, a_4 \between a_1,\, -b_0)^4 = \rho_\xi^4,$$

and, as before, $\rho_\xi = \alpha_2 a_1 - \alpha_3 b_0 = (\alpha_2\beta_3)\,\beta_1^3$, so that

$$v_4 = (\alpha_2\beta_3)\,\beta_1^3 \cdot (\alpha_2\gamma_3)\,\gamma_1^3 \cdot (\alpha_2\delta_3)\,\delta_1^3 \cdot (\alpha_2\varepsilon_3)\,\varepsilon_1^3,$$

and therefore

$$U_4 = (\alpha\beta u)(\alpha\gamma u)(\alpha\delta u)(\alpha\varepsilon u)\,\beta_x^3\gamma_x^3\delta_x^3\varepsilon_x^3$$
$$= \chi_8 x_1^{12}u_1^4 + \dots \qquad (28).$$

Next, h_4 is the Hessian of v_4, and therefore

$$h_4 = \frac{1}{2}\,(\rho\sigma)^2\rho_\xi^2\sigma_\xi^2$$
$$= \frac{1}{2}\,(\alpha_2\beta_3)^2 \cdot (\alpha_2\gamma_3)\,\gamma_1^3(\alpha_2\delta_3)\,\delta_1^3 \cdot (\beta_2\lambda_3)\,\lambda_1^3\,(\beta_2\mu_3)\,\mu_1^3,$$

hence

$$H_4 = \frac{1}{2}\,(\alpha\beta u)^2(\alpha\gamma u)(\alpha\delta u)(\beta\lambda u)(\beta\mu u)\,\gamma_x^3\delta_x^3\lambda_x^3\mu_x^3$$
$$= \chi_9 x_1^{12}u_1^6 + \dots \qquad (29).$$

Again, for ϕ_4, which is the cubicovariant of v_4, we have

$$\phi_4 = \frac{1}{2}\,(\rho\sigma)^2(\sigma\tau)\,\rho_\xi^2\sigma_\xi\tau_\xi^3$$
$$= \frac{1}{2}\,(\alpha_2\beta_3)^2(\beta_2\gamma_3) \cdot (\alpha_2\delta_3)\,\delta_1^3\,(\alpha_2\varepsilon_3)\,\varepsilon_1^3 \cdot (\beta_2\theta_3)\,\theta_1^3 \cdot (\gamma_2\lambda_3)\,\lambda_1^3\,(\gamma_2\mu_3)\,\mu_1^3\,(\gamma_2\nu_3)\,\nu_1^3,$$

and therefore

$$\Phi_4 = \frac{1}{2}\,(\alpha\beta u)^2(\beta\gamma u)(\alpha\delta u)(\alpha\varepsilon u)(\beta\theta u)(\gamma\lambda u)(\gamma\mu u)(\gamma\nu u)\,\delta_x^3\varepsilon_x^3\theta_x^3\lambda_x^3\mu_x^3\nu_x^3$$
$$= \chi_{10} x_1^{18}u_1^9 + \dots \qquad (30).$$

For i_4, the quadrinvariant of v_4, we have

$$i_4 = \frac{1}{2}\,(\rho\sigma)^4,$$

and therefore

$$I_4 = \frac{1}{2}\,(\alpha\beta u)^4$$
$$= \chi_{11} u_1^4 + \dots \qquad (31).$$

Lastly, j_{24} is the Jacobian of v_2 and v_4, so that

$$\begin{aligned} j_{24} &= \alpha_1^2 (\epsilon\rho)\, \epsilon_\xi \rho_\xi^3 \\ &= \alpha_1^2 (\alpha_2\beta_3) . (\alpha_2\gamma_3)\, \gamma_1^3 . (\beta_2\lambda_3)\, \lambda_1^3 (\beta_2\mu_3)\, \mu_1^3 (\beta_2\nu_3)\, \nu_1^3, \end{aligned}$$

and therefore

$$\begin{aligned} J_{24} &= (\alpha\beta u)(\alpha\gamma u)(\beta\lambda u)(\beta\mu u)(\beta\nu u)\, \alpha_x^2\gamma_x^3\lambda_x^3\mu_x^3\nu_x^3 \\ &= \chi_{12} x_1^{14} u_1^5 + \dots . \end{aligned} \qquad (32).$$

It will be seen that in every case the value of $m - p$ agrees with the required value in the earlier table, and we are therefore now justified in enunciating the following theorem:

Every concomitant of the ternary quartic can be algebraically expressed in terms of u_x (the universal concomitant) and of the twelve fundamental concomitants U_1 (the quartic itself); H_2, U_2; U_3, H_3, Φ_3, J_{23}; U_4, H_4, Φ_4, I_4, J_{24}; the leading coefficients of these concomitants are given in (13) and (21), *their order and class in* (22) *to* (32), *and their full expressions can be obtained by* (I) and (II).

41. This fundamental system may be modified—as in §34 for the cubic—in the case of all the concomitants which have Jacobian functions in the leading coefficients. We have

$$\phi_3^2 = f v_3^2 - 4h_3^3,$$

where f is the discriminant of v_3; and thus we may replace ϕ_3 by f, the symbolic form of which is

$$(\alpha_2\beta_3)^2(\gamma_2\delta_3)^2(\alpha_2\gamma_3)(\beta_2\delta_3)\,\alpha_1\beta_1\gamma_1\delta_1,$$

and therefore

$$\begin{aligned} F &= (\alpha\beta u)^2(\gamma\delta u)^2(\alpha\gamma u)(\beta\delta u)\,\alpha_x\beta_x\gamma_x\delta_x \\ &= (a_3^2 d_0^2 - 6a_3b_2c_1d_0 + 4a_3c_1^3 + 4d_0b_2^3 - 3b_2^2c_1^2)\, x_1^4 u_1^6 + \dots \end{aligned} \qquad (26')$$

will *replace* Φ_3 *in the system.*

Similarly we might replace j_{23} by p, the intermediate invariant of v_2 and h_3; its symbolic form is

$$\tfrac{1}{2}\, \alpha_1^2 (\beta_2\gamma_3)^2(\alpha_2\gamma_3)(\alpha_2\beta_3)\,\beta_1\gamma_1,$$

and therefore we have a function

$$\begin{aligned} &\tfrac{1}{2} (\beta\gamma u)^2(\alpha\gamma u)(\alpha\beta u)\,\alpha_x^2\beta_x\gamma_x \\ &= \tfrac{1}{6} (\alpha\beta\gamma)(\alpha\beta u)(\alpha\gamma u)(\beta\gamma u)\, u_x\alpha_x\beta_x\gamma_x, \end{aligned}$$

so that
$$P = \frac{1}{6}(\alpha\beta\gamma)(\alpha\beta u)(\alpha\gamma u)(\beta\gamma u)\,\alpha_x\beta_x\gamma_x$$
$$= \{c_0(a_3c_1 - b_2^2) - b_1(a_3d_0 - b_2c_1) + a_2(b_2d_0 - c_1^2)\}x_1^3u_1^3 + \ldots \quad (27')$$
will *replace* J_{23} *in the system.*

Next, ϕ_4 being the cubicovariant of v_4, we have
$$\phi_4^2 + j_4v_4^3 - i_4h_4v_4^2 + 4h_4^3 = 0,$$
where j_4 is the cubinvariant of v_4; and so we may replace ϕ_4 by j_4, the symbolical expression of which is
$$\frac{1}{6}(\rho\sigma)^2(\sigma\tau)^2(\rho\tau)^2$$
$$= \frac{1}{6}(\alpha_2\beta_3)^2(\beta_2\gamma_3)^2(\alpha_2\gamma_3)^2,$$
so that
$$J_4 = \frac{1}{6}(\alpha\beta u)^2(\beta\gamma u)^2(\alpha\gamma u)^2$$
$$= (e_0b_2a_4 + 2d_1b_2c_3 - b_2^3 - e_0c_3^2 - d_1^2a_4)\,u_1^6 + \ldots \quad (30')$$
will *replace* Φ_4 *in the system.*

Lastly, since j_{24} is the Jacobian of v_2 and v_4, and we retain h_2 and h_4, it can be replaced by the second transvectant of v_2 and v_4, the symbolical expression of which is
$$q = \frac{1}{2}\alpha_1^2(\varepsilon\rho)^2\rho_\xi^2$$
$$= \frac{1}{2}\alpha_1^2(\alpha_2\beta_3)^2.(\beta_2\gamma_3)\gamma_1^3(\beta_2\delta_3)\delta_1^3,$$
and therefore
$$Q = \frac{1}{2}(\alpha\beta u)^2(\beta\gamma u)(\beta\delta u)\,\alpha_x^2\gamma_x^2\delta_x^2$$
$$= \{(a_2e_0 - 2b_1d_1 + c_0c_2)a_1^2 - 2(a_2d_1 - 2b_1c_2 + b_3c_0)a_1b_0$$
$$+ (a_2c_2 - 2b_1b_3 + c_0a_4)b_0^2\}x_1^6u_1^4 + \ldots \quad (32')$$
will *replace* J_{24} *in the system.*

42. As illustrations of the general theorem of §40, the following may be taken. It has been shown by Maisano* that all the concomitants of the quartic of the second degree are (l. c., p. 201) $a_x^2b_x^2(abu)^2$, which is effectively H_2 of (22), and $(abu)^4$, which is I_4 of (31); and that among those of the third degree are

* "Sistemi completi dei primi cinque gradi della forma ternaria biquadratica e degl' invarianti, covarianti e contravarianti di sesto grado," Batt. Giorn. di Mat., t. XIX (1881), pp. 198–237.

(l. c., p. 203) $(abu)^2(bcu)^2(cau)^2$, which is effectively J_4 of (30′), and $(abc)^2a_x^2b_x^2c_x^2$, which is $U_2 - U_1H_2$. Another concomitant of this degree is

$$\Theta = a_xb_x^2c_x^3\,(abu)^2(acu) = \{a_1\,(a_2d_0 - 2b_1c_1 + b_2c_0) - b_0\,(c_1a_2 - 2b_2c_1 + a_3c_0)\}\,x_1^6u_1^3 + \ldots,$$

so that
$$\Theta U_2U_3 = H_2U_3^2 + H_3U_2^2 + J_{23}^2.$$

Another is

$$\begin{aligned}\Psi &= (abu)^3(abc)\,c_x^3\\ &= [a_0\,(e_0a_4 - 4d_1b_3 + 3c_2^2) + a_1\,(d_0b_3 - 3c_2c_1 + 3b_2d_1 - a_3e_0)\\ &\qquad\qquad - b_0\,(d_0a_4 - 3b_3c_1 + 3b_2c_2 - a_3d_1)]\,u_1^3x_1^3 + \ldots.\\ &= \psi u_1^3x_1^3 + \ldots,\end{aligned}$$

and it is not difficult to prove that

$$\begin{aligned}&(u_x\Psi - U_1I_4)\,U_2^3U_3^2U_4^2\\ &= U_2^3U_3^3\Phi_4 - U_2^3U_4^3\Phi_3 + 3U_2^2\,(H_4U_3^2 + H_3U_4^2)\,(U_4J_{23} - U_3J_{24}) + (U_4J_{23} - U_3J_{24})^3.\end{aligned}$$

Lastly, when the tabulated value of $A = \frac{1}{6}\,(abc)^4$ is taken as calculated by Bernardi, it can be arranged in the form

$$A = \psi - 12p + 3N,$$

where p is the coefficient of (27′), ψ is the coefficient just given and

$$N = e_0a_2^2 - 4d_1a_2b_1 + 2c_2\,(c_0a_2 + 2b_1^2) - 4b_3c_0b_1 + a_4c_0^2,$$

evidently a simultaneous invariant of v_4 and v_2 and expressible in terms of v_2, h_2, v_4, h_4, i_4, ϕ_4, j_{24}.

The general method of expressing any concomitant in terms of the set, here proved to be complete, is to take its leading coefficient ϑ, which must be a simultaneous concomitant of v_0, v_2, v_3, v_4 and must be expressible in terms of the quantities in (13) and (21). Moreover, since they are binariants, it is sufficient to consider the coefficients of the highest powers of a_1 contained by them; and it is found that in every case ϑ can be arranged as combinations of quantities, which are concomitants in a_1 and b_0. Thus, for instance, Θ above has for its leading coefficient the simultaneous linear covariant called L_1 by Salmon (p. 178); Ψ has a leading coefficient composed of a part v_0i_4 and the simultaneous covariant called L_2 by Salmon (p. 179), and similarly for A.

IV.—*Complete System of Algebraically Independent Concomitants for the n^{tic}.*

43. It is at once evident that all the leading coefficients of the concomitants just obtained for the quartic consist of (i) the algebraically independent invariants and covariants of the binary quadratic, the binary cubic and the binary quartic in a_1 and $-b_0$ as variables; of (ii) the Jacobians of this binary quadratic and binary cubic, and this binary quadratic and binary quartic; and (iii) of the original quantic.

The forms of the characteristic differential equations satisfied by these leading coefficients show that every solution is a concomitant of the simultaneous system of binary quantics formed in the above way; the theory shows that every such solution can be algebraically expressed in terms of the members of the above set.

And the result is true for the general quantic of order n, so that we can now state a complete system of algebraically independent concomitants.

44. First, let $U = \alpha_x^n$ be the quantic which, written explicitly, takes the form

$$
\begin{aligned}
(a_0, a_1, \ldots, a_n \between x_1, x_3)^n &+ \frac{n!}{\overline{n-1}!\,1!} x_2 (b_0, b_1, \ldots, b_{n-1} \between x_1, x_3)^{n-1} \\
&+ \frac{n!}{\overline{n-2}!\,2!} x_2^2 (c_0, c_1, \ldots, c_{n-2} \between x_1, x_3)^{n-2} \\
&+ \frac{n!}{\overline{n-3}!\,3!} x_2^3 (d_0, d_1, \ldots, d_{n-2} \between x_1, x_3)^{n-3} + \ldots.
\end{aligned}
$$

We shall represent the leading coefficients in symbolical forms, as in §§33, 40; their explicit forms are obtainable in the same way as the explicit forms of concomitants of binary quantics, and indeed are the same as those binariants when a_1 and $-b_0$ replace the variables.

We have, first

$$U = \alpha_x^n = a_0 x_1^n + \ldots.$$

Next, let

$$v_2 = (c_0, b_1, a_2 \between a_1, -b_0)^2 = \alpha_1^{n-2} (\alpha_2 a_1 - \alpha_3 b_0)^2 = \alpha_1^{n-2} \rho_\xi^2,$$

then the concomitant is

$$U_2 = v_2 x_1^{3n-4} u_1^2 + \ldots;$$

and $h_2 = a_2 c_0 - b_1^2$, the Hessian of v_2, so that the concomitant is

$$H_2 = h_2 x_1^{2n-4} u_1^2 + \ldots,$$

and as usual we have

$$(U_2 - UH_2) \div u_x^2 = \text{Hessian of } U.$$

Next, let

$$v_3 = (d_0, c_1, b_2, a_3 \between a_1, -b_0)^3 = \alpha_1^{n-3}\rho_\xi^3,$$

then the concomitant is

$$U_3 = v_3 x_1^{4n-6} u_1^3 + \ldots . ;$$

and the associated set is given by h_3 (the Hessian) and ϕ_3 (the cubicovariant) of v_3, the concomitants being

$$H_3 = h_3 x_1^{4n-8} u_1^4 + \ldots . ,$$
$$\Phi_3 = \phi_3 x_1^{6n-12} u_1^6 + \ldots . ,$$

and so on.

In general, let

$$v_r = (\ldots . , c_{r-2}, b_{r-1}, a_r \between a_1, -b_0)^r = \alpha^{n-r}\rho_\xi^r,$$

where $\rho_\xi = \alpha_2 a_1 - \alpha_3 b_0$; and we shall suppose ρ, σ, τ to be equivalent symbols. Then it is known from the theory of binary quantics that all the concomitants can be expressed in terms of the following set of binariants of the second and of the third degrees alternately in $\ldots . , c_{r-2}, b_{r-1}, a_r$, viz.

$$\omega(2, r) = \alpha_1^{n-r}\beta_1^{n-r}(\rho\sigma)^2 \rho_\xi^{r-2}\sigma_\xi^{r-2},$$
$$\omega(3, r) = \alpha_1^{n-r}\beta_1^{n-r}\gamma_1^{n-r}(\rho\sigma)^2(\sigma\tau)\rho_\xi^{r-2}\sigma_\xi^{r-3}\tau_\xi^{r-1},$$
$$\omega(4, r) = \alpha_1^{n-r}\beta_1^{n-r}(\rho\sigma)^4 \rho_\xi^{r-4}\sigma_\xi^{r-4},$$
$$\omega(5, r) = \alpha_1^{n-r}\beta_1^{n-r}\gamma_1^{n-r}(\rho\sigma)^4(\sigma\tau)\rho_\xi^{r-4}\sigma_\xi^{r-5}\tau_\xi^{r-1},$$
$$\omega(6, r) = \alpha_1^{n-r}\beta_1^{n-r}(\rho\sigma)^6 \rho_\xi^{r-6}\sigma_\xi^{r-6},$$
$$\omega(7, r) = \alpha_1^{n-r}\beta_1^{n-r}\gamma_1^{n-r}(\rho\sigma)^6(\sigma\tau)\rho_\xi^{r-6}\sigma_\xi^{r-7}\tau_\xi^{r-1},$$

and so on; the symbols ρ_ξ, σ_ξ, τ_ξ in these respectively denote $\alpha_2 a_1 - \alpha_3 b_0$, $\beta_2 a_1 - \beta_3 b_0$, and $\gamma_2 a_1 - \gamma_3 b_0$. The series of functions concludes with the term $\omega(r, r)$, the form of which depends on the evenness or oddness of r.

In order to find the order and the class for each of the concomitants, we must take two separate typical forms, say $\omega(2s, r)$ and $\omega(2s+1, r)$, where r may not be less than $2s$ in the former nor than $2s+1$ in the latter.

For the former we have

$$\omega(2s, r) = \alpha_1^{n-r}\beta_1^{n-r}(\rho\sigma)^{2s}\rho_\xi^{r-2s}\sigma_\xi^{r-2s},$$

when this is changed into a further symbolical form for the concomitant, $(\rho\sigma)$

becomes $(\alpha_2\beta_3)$ and so ultimately comes to be $(\alpha\beta u)$; that is, every power of $(\rho\sigma)$ introduces a unit for the class. Again, ρ_ξ becomes of the form

$$\theta_1^{n-1}(\beta_2\theta_3 - \beta_3\theta_2) = \theta_1^{n-1}(\beta_2\theta_3),$$

and so ultimately comes to be $\theta_x^{n-1}(\beta\theta u)$; that is, every factor of the form ρ_ξ introduces a unit for the class and $n-1$ for the order. And α_1^{n-r} ultimately comes to be α_x^{n-r}, and so with β_x. Hence finally, the order is

$$2(n-r) + 2(r-2s)(n-1) = 2n(r-2s+1) - 4(r-s),$$

and the class is $$2s + 2(r-2s) = 2(r-s),$$

and therefore the concomitant is

$$W_{2s,\,r} = \omega(2s,\, r)\, x_1^{2n(r-2s+1)-4(r-s)} u_1^{2(r-s)} + \ldots.$$

Similarly for $\omega(2s+1,\, r)$, the symbol for which is

$$\alpha_1^{n-r}\beta_1^{n-r}\gamma_1^{n-r}(\rho\sigma)^{2s}(\sigma\tau)\,\rho_\xi^{r-2s}\sigma_\xi^{r-2s-1}\tau_\xi^{r-1},$$

we have

$$W_{2s+1,\,r} = \omega(2s+1,\, r)\, x_1^{n(3r-4s+1)-2(3r-2s-1)} u_1^{3r-2s-1} + \ldots.$$

For this class of the complete set of the concomitants given by $W_{\mu,\,r}$, the values of μ, for a given value of r, are $0, 2, 3, \ldots, r$ and $W_{0,\,r}$ has for its leading coefficient v_r, being

$$= v_r x_1^{n(r+1)-2r} u_1^r + \ldots;$$

and the values of r are $2, 3, \ldots, n$. Thus the total number of concomitants in this division is $\frac{1}{2}n(n+1) - 1$.

45. Next, for the class of concomitants whose coefficients are the algebraically independent Jacobians of $v_2, v_3, \ldots, v_n$, we take $j_{2,\,3}, j_{2,\,4}, j_{2,\,5}, \ldots, j_{2,\,n}$. Evidently

$$j_{2,\,r} = \alpha_1^{n-2}\beta_1^{n-r}(\rho\sigma)\,\rho_\xi\sigma_\xi^{r-1},$$

where $\rho_\xi = \alpha_2 a_1 - \alpha_3 b_0$, $\sigma_\xi = \beta_2 a_1 - \beta_3 b_0$; and the concomitant is

$$J_{2,\,r} = j_{2,\,r}\, x_1^{n(r+2)-2r-2} u_1^{r+1} + \ldots.$$

The values of r are $3, 4, \ldots, n$, and the total number in this class of concomitants is therefore $n-2$.

Hence the *total number of concomitants* is

1, for the original quantic,

$+\frac{1}{2}n(n+1)-1$ for those in the first of the classes,

$+n-2$ " " " second " "

i. e. the total number is $\frac{1}{2}(n+4)(n-1)$, agreeing with the former result.

These concomitants are algebraically independent of one another, and every concomitant of the quantic can be algebraically expressed in terms of them.

V.—*System of Two Quadratics.*

46. The two quadratics may be taken in the forms

$$a_0x_1^2+2b_0x_1x_2+2a_1x_1x_3+c_0x_2^2+2b_1x_2x_3+a_2x_3^2,$$
$$a_0'x_1^2+2b_0'x_1x_2+2a_1'x_1x_3+c_0'x_2^2+2b_1'x_2x_3+a_2'x_3^2;$$

the characteristic equations are

$$D_1+D_1'=a_1\frac{\partial}{\partial b_0}+a_2\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial c_0}+a_1'\frac{\partial}{\partial b_0'}+a_2'\frac{\partial}{\partial b_1'}+2b_1'\frac{\partial}{\partial c_0'},$$
$$\Delta=D_6+D_6'=b_0\frac{\partial}{\partial a_1}+c_0\frac{\partial}{\partial b_1}+2b_1\frac{\partial}{\partial a_2}+b_0'\frac{\partial}{\partial a_1'}+c_0'\frac{\partial}{\partial b_1'}+2b_1'\frac{\partial}{\partial a_2'}.$$

There are twelve coefficients in all; there will therefore be eleven equations subsidiary to, and eleven independent solutions of, $D_1+D_1'=0$; and ultimately there will be (§§18 and 35) nine independent solutions common to the two equations.

From the form of the characteristic equations, it at once follows that they are the *simultaneous concomitants of two binary quadratics, the literal coefficients of which are* c_0, b_1, a_2, *and* c_0', b_1', a_2', *and that the variables of the concomitants are two sets, viz.* a_1 *and* $-b_0$, a_1' *and* $-b_0'$.

47. The subsidiary equations for $D_1+D_1'=0$ are

$$\frac{da_0}{0}=\frac{da_1}{0}=\frac{da_2}{0}=\frac{da_0'}{0}=\frac{da_1'}{0}=\frac{da_2'}{0}=\frac{db_0}{a_1}=\frac{db_1}{a_2}=\frac{dc_0}{2b_1}=\frac{db_0'}{a_1'}=\frac{db_1'}{a_2'}=\frac{dc_0'}{2b_1'},$$

of which six integrals are immediately given by

$$\theta_0=a_0,\quad \theta_1=a_1,\quad \theta_2=a_2;$$
$$\theta_0'=a_0',\quad \theta_1'=a_1',\quad \theta_2'=a_2',$$

and we may take either θ_1 or θ_1' as a "variable of reference."

The full system of equations, subsidiary to the solution of $\Delta = 0$ in functional combinations of the solutions of $D_1 + D_1' = 0$, are for the alternative variables of reference:

$$\left.\begin{array}{l} \theta_1\Delta\theta_2 - 2\theta_2\Delta\theta_1 = 2\theta_3 \\ \theta_1\Delta\theta_3 - \theta_3\Delta\theta_1 = \theta_4 \\ \theta_1\Delta\theta_1' - \theta_1'\Delta\theta_1 = \phi \\ \theta_1\Delta\theta_2' - 2\theta_2'\Delta\theta_1 = 2\psi_3 \\ \theta_1\Delta\psi_3 - \psi_3\Delta\theta_1 = \psi_4 \\ \hline \theta_1\Delta\theta_3' - \theta_3'\Delta\theta_1 = \mu_4 \\ \theta_1\Delta\chi_3 - \chi_3\Delta\theta_1 = \lambda_4 \end{array}\right\} \qquad \left.\begin{array}{l} \theta_1'\Delta\theta_2' - 2\theta_2'\Delta\theta_1' = 2\theta_3' \\ \theta_1'\Delta\theta_3' - \theta_3'\Delta\theta_1' = \theta_4' \\ \theta_1'\Delta\theta_1 - \theta_1\Delta\theta_1' = -\phi \\ \theta_1'\Delta\theta_2 - 2\theta_2\Delta\theta_1' = 2\chi_3 \\ \theta_1'\Delta\chi_3 - \chi_3\Delta\theta_1' = \chi_4 \\ \hline \theta_1'\Delta\theta_3 - \theta_3\Delta\theta_1' = \lambda_4 \\ \theta_1'\Delta\psi_3 - \psi_3\Delta\theta_1' = \mu_4 \end{array}\right\}$$

where the eleven quantities defined by the equations

$$\left.\begin{array}{l} \theta_3 = b_1a_1 - a_2b_0 \\ \chi_3 = b_1a_1' - a_2b_0' \end{array}\right\}, \quad \left.\begin{array}{l} \psi_3 = b_1'a_1 - a_2'b_0 \\ \theta_3' = b_1'a_1' - a_2'b_0' \end{array}\right\}, \quad \phi = a_1b_0' - a_1'b_0,$$

$$\left.\begin{array}{l} \theta_4 = c_0a_1^2 - 2b_1a_1b_0 + a_2b_0^2 \\ \lambda_4 = c_0a_1a_1' - b_1(a_1b_0' + a_1'b_0) + a_2b_0b_0' \\ \chi_4 = c_0a_1'^2 - 2b_1a_1'b_0' + a_2b_0'^2 \end{array}\right\}, \quad \left.\begin{array}{l} \psi_4 = c_0'a_1^2 - 2b_1'a_1b_0 + a_2'b_0^2 \\ \mu_4 = c_0'a_1a_1' - b_1'(a_1b_0' + a_1'b_0) + a_2'b_0b_0' \\ \theta_4' = c_0'a_1'^2 - 2b_1'a_1'b_0' + a_2'b_0'^2 \end{array}\right\},$$

all are solutions of $D_1 + D_1' = 0$. Further, the quantities $\theta_4, \psi_4; \theta_4', \chi_4; \phi; \lambda_4, \mu_4$, are solutions also of $\Delta = 0$.

For each of the variables of reference, the first five of the equations of the set are sufficient to give all the equations, subsidiary to $\Delta = 0$ and necessary for the derivation of solutions additional to those already obtained.

Taking θ_1 as the variable of reference, we have common solutions of the two characteristic equations given by

$$\theta_0, \theta_0'; \theta_4, \psi_4; \phi;$$

and four more are necessary, given by the solutions of the first five equations in the first bracket of modified equations subsidiary to $\Delta = 0$. If, then, we substitute

$$\left.\begin{array}{l} \dfrac{\theta_2}{\theta_1^2} = p \\ \dfrac{\theta_2'}{\theta_1'^2} = p' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\theta_3}{\theta_1} = q \\ \dfrac{\theta_3'}{\theta_1'} = q' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\theta_2'}{\theta_1^2} = r \\ \dfrac{\theta_2}{\theta_1'^2} = r' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\psi_3}{\theta_1} = \rho \\ \dfrac{\chi_3}{\theta_1'} = \rho' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\chi_3}{\theta_1} = \sigma \\ \dfrac{\psi_3}{\theta_1'} = \sigma' \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{\theta_3'}{\theta_1} = s \\ \dfrac{\theta_3}{\theta_1'} = s' \end{array}\right\}; \quad \frac{\theta_1'}{\theta_1} = \varepsilon,$$

then the equations come to be

$$\left.\begin{aligned} \theta_1^2\Delta p &= 2q \\ \theta_1^2\Delta q &= \theta_4 \\ \theta_1^2\Delta \varepsilon &= \phi \\ \theta_1^2\Delta r &= 2\rho \\ \theta_1^2\Delta \rho &= \psi_4 \\ \hline \theta_1^2\Delta s &= \mu_4 \\ \theta_1^2\Delta \sigma &= \lambda_4 \end{aligned}\right\}, \qquad \left.\begin{aligned} \theta_1'^2\Delta p' &= 2q' \\ \theta_1'^2\Delta q' &= \theta_4' \\ \theta_1'^2\Delta \varepsilon^{-1} &= -\phi \\ \theta_1'^2\Delta r' &= 2\rho' \\ \theta_1'^2\Delta \rho' &= \chi_4 \\ \hline \theta_1'^2\Delta s' &= \lambda_4 \\ \theta_1'^2\Delta \sigma' &= \mu_4 \end{aligned}\right\},$$

and what we wish are four independent solutions of the first five equations in the former of these brackets. Such solutions are

$$\begin{aligned} \vartheta_2 &= p\theta_4 - q^2 = a_2c_0 - b_1^2, \\ B &= \varepsilon\theta_4 - q\phi = \lambda_4, \\ \vartheta_2' &= r\psi_4 - \rho^2 = a_2'c_0' - b_1'^2, \\ C &= \varepsilon\psi_4 - \rho\phi = \mu_4; \end{aligned}$$

and λ_4 and μ_4 are the respective Jacobians of ϕ and θ_4 (with a_1 and b_0 as variables) and of ϕ and ψ_4 (with a_1 and b_0 as variables).

Hence it follows that *every common solution of the two characteristic equations can be expressed in terms of the nine common solutions*

$$\theta_0,\ \theta_4,\ \vartheta_2;\ \theta_0',\ \psi_4,\ \vartheta_2';\ \phi,\ \lambda_4,\ \mu_4.$$

48. If we take the first five of the modified equations in the second bracket, we find the four new solutions to be

$$\begin{aligned} \vartheta_2 &= r'\chi_4 - \rho'^2 = a_2c_0 - b_1^2, \\ B &= q'\phi + \theta_4'\varepsilon^{-1} = \lambda_4, \\ \vartheta_2' &= p'\theta_4' - q'^2 = a_2'c_0' - b_1'^2, \\ C &= \rho'\phi + \chi_4\varepsilon^{-1} = \mu_4; \end{aligned}$$

and *every solution can be expressed in terms of the set*

$$\theta_0,\ \vartheta_2,\ \chi_4;\ \theta_0',\ \theta_4',\ \vartheta_2';\ \phi,\ \lambda_4,\ \mu_4.$$

49. Other solutions of the system of equations are

$$f_{12} = a_2c_0' + a_2'c_0 - 2b_1b_1',$$

an intermediate between ϑ_2 and ϑ_2';

$$g = \quad \rho\theta_4 - q\psi_4 = (b_1'c_0 - b_1c_0')\,a_1^2 - a_1b_0\,(a_2'c_0 - a_2c_0') + b_0^2\,(a_2'b_1 - a_2b_1'),$$
$$g' = -\rho'\theta_4' + q'\chi_4 = (b_1'c_0 - b_1c_0')\,a_1'^2 - a_1'b_0'\,(a_2'c_0 - a_2c_0') + b_0'^2\,(a_2'b_1 - a_2b_1'),$$

the former a Jacobian of θ_4 and ψ_4 (in a_1 and b_0 as variables), the latter a Jacobian of θ_4' and χ_4 (in a_1' and b_0' as variables);

$$g_{12} = s\theta_4 - q\mu_4 = (b_1'c_0 - b_1c_0')\,a_1a_1' - a_1b_0'\,(a_2'c_0 - b_1b_1') - a_1'b_0\,(b_1b_1' - a_2c_0') + b_0b_0'\,(a_2'b_1 - a_2b_1'),$$

an intermediary between g and g', and a Jacobian of θ_4 and μ_4.

These four are the most important of the solutions, and they will be used in connection with the fundamental system to be made symmetrical later.

Other solutions—the simplest in form—are as follows: they should be expressible in terms of the fundamental set, and the verification of this leads to the values given for them. The left-hand sides of the equations give the solutions, the right-hand their values:

$$\begin{array}{llll}
s\psi_4 - \rho\mu_4 = -\phi\vartheta_2', & s'\chi_4 - \rho'\lambda_4 = \phi\vartheta_2; \\
q\lambda_4 - \sigma\theta_4 = \phi\vartheta_2, & q'\mu_4 - \sigma'\theta_4' = -\phi\vartheta_2'; \\
s\theta_4 - q\mu_4 = g_{12}, & s'\theta_4' - q'\lambda_4 = -g_{12}; \\
\rho\theta_4 - q\psi_4 = g, & \rho'\theta_4' - q'\chi_4 = -g'; \\
\varepsilon\theta_4 - q\phi = \lambda_4, & \dfrac{1}{\varepsilon}\theta_4' + q'\phi = \mu_4; \\
\varepsilon\psi_4 - \rho\phi = \mu_4, & \dfrac{1}{\varepsilon}\chi_4 + \rho'\phi = \lambda_4; \\
\varepsilon\mu_4 - s\phi = \theta_4, & \dfrac{1}{\varepsilon}\lambda_4 + s'\phi = \theta_4'; \\
\varepsilon\lambda_4 - \sigma\phi = \chi_4, & \dfrac{1}{\varepsilon}\mu_4 + \sigma'\phi = \psi_4; \\
\rho\lambda_4 - \sigma\psi_4 = g_{12} + \phi f_{12}, & \rho'\mu_4 - \sigma'\chi_4 = -g_{12} - \phi f_{12}; \\
s\lambda_4 - \sigma\mu_4 = g, & s'\mu_4 - \sigma'\lambda_4 = -g'.
\end{array}$$

And the equations which express the values of the quantities g, g', g_{12}, f_{12}, ... in terms of the fundamental systems are

$$\begin{aligned}
\theta_4'\psi_4 &= \mu_4^2 + \phi^2\vartheta_2', \\
\theta_4\chi_4 &= \lambda_4^2 + \phi^2\vartheta_2, \\
g\phi &= \lambda_4\psi_4 - \mu_4\theta_4, \\
g'\phi &= \mu_4\chi_4 - \lambda_4\theta_4', \\
&\left.\begin{aligned} g^2 &= -\vartheta_2\psi_4^2 + f_{12}\theta_4\psi_4 - \vartheta_2'\theta_4^2 \\ g'^2 &= -\vartheta_2'\chi_4^2 + f_{12}\theta_4'\chi_4 - \vartheta_2\theta_4'^2 \end{aligned}\right\}, \\
&\left.\begin{aligned} g_{12}\psi_4 &= \mu_4 g - \theta_4\vartheta_2'\phi \\ g_{12}\chi_4 &= \lambda_4 g' - \theta_4'\vartheta_2\phi \end{aligned}\right\}.
\end{aligned}$$

It may be remarked that a form more directly intermediate between g and g' is given by $g_{12}+\frac{1}{2}f_{12}\phi$, the value of which is

$$(b_1'c_0-b_1c_0')\,a_1a_1'-\frac{1}{2}(a_2'c_0-a_2c_0')(a_1b_0'+a_1'b_0)+(a_2'b_1-a_2b_1')\,b_0b_0',$$

which with similar forms will be adopted for the system in the case of three quadratics. The form g_{12} adopted in the present system is directly connected with one of Gordan's concomitants, and the corresponding concomitant has its order and its class each greater by unity than those of the present g_{12}.

50. The fundamental system can be modified so as to be symmetrical with regard to the two quantics. We have seen that

$$\theta_4'\psi_4=\mu_4^2+\phi^2\vartheta_2',$$
$$\theta_4\chi_4=\lambda_4^2+\phi^2\vartheta_2,$$

so that in the first fundamental system we can replace λ_4 and μ_4 by χ_4 and θ_4' respectively, and in the second by θ_4 and ψ_4 respectively. The two systems are the same, and it thus follows that *every common solution can be expressed in terms of*

$$\theta_0,\ \theta_0';\ \phi;\ \vartheta_2,\ \vartheta_2';\ \theta_4,\ \psi_4,\ \chi_4,\ \theta_4'.$$

51. It is now necessary to determine the order and the grade of each of the concomitants determined by these leading coefficients. It is easy to show that

$$\begin{aligned}
U&=\theta_0x_1^2+\dots,\\
U'&=\theta_0'x_1^2+\dots,\\
\Phi&=\phi x_1^2u_1+\dots,\\
\Theta_2&=\vartheta_2u_1^2+\dots,\\
\Theta_2'&=\vartheta_2'u_1^2+\dots,\\
\Theta_4&=\theta_4x_1^2u_1^2+\dots,\\
\Psi_4&=\psi_4x_1^2u_1^2+\dots,\\
\mathrm{X}_4&=\chi_4x_1^2u_1^2+\dots,\\
\Theta_4'&=\theta_4'x_1^2u_1^2+\dots,
\end{aligned}$$

and therefore *every simultaneous concomitant of the two quadratics can be expressed algebraically in terms of* U, U', Φ, Θ_2, Θ_2', Θ_4, Ψ_4, X_4, Θ_4'.

In addition to these nine, it is convenient to have other six, the leading coefficients of which are respectively λ_4, μ_4; g, g', g_{12}, and f_{12}. It is easy to determine their order and class; the concomitants are

$$\begin{aligned}
\Lambda_4 &= \lambda_4 x_1^2 u_1^2 + \ldots\ldots, \\
\mathrm{M}_4 &= \mu_4 x_1^2 u_1^2 + \ldots\ldots, \\
G &= g x_1^2 u_1^3 + \ldots\ldots, \\
G' &= g' x_1^2 u_1^3 + \ldots\ldots, \\
G_{12} &= g_{12} x_1 u_1^2 + \ldots\ldots, \\
F_{12} &= f_{12} u_1^2 \ + \ldots\ldots,
\end{aligned}$$

and the equations which give the values of these in terms of the members of the fundamental system are

$$\begin{aligned}
\mathrm{M}_4^2 &= \Theta_4'\Psi_4 - \Phi^2\Theta_2', \\
\Lambda_4^2 &= \Theta_4 X_4 - \Phi^2\Theta_2, \\
G\Phi &= \mathrm{M}_4\Theta_4 - \Lambda_4\Psi_4, \\
G'\Phi &= \Lambda_4\Theta_4' - \mathrm{M}_4 X_4, \\
&\left.\begin{aligned}
\Theta_4\Psi_4F_{12} &= G^2 \quad + \Theta_2\Psi_4^2 + \Theta_2'\Theta_4^2 \\
\Theta_4'X_4F_{12} &= G_1'^2 \quad + \Theta_2\Theta_4'^2 + \Theta_2'X_4^2
\end{aligned}\right\}, \\
&\left.\begin{aligned}
u_x\Psi_4G_{12} &= \mathrm{M}_4 G - \Theta_4\Theta_2'\Phi \\
u_x X_4 G_{12} &= \Lambda_4 G' - \Theta_4'\Theta_2\Phi
\end{aligned}\right\}.
\end{aligned}$$

The six concomitants Λ_4, M_4, G, G', G_{12}, F_{12} may be used as subsidiary to the symmetrical set, it being understood that in expressions they represent the foregoing combinations of the members of that set.

52. Now Gordan has shown* that the number of asyzygetic concomitants of a system of two ternary quadratics is 20, and he has given (l. c.) the symbolical expressions for them. From the foregoing theory it follows that each of them must be expressible in terms of the set of nine above obtained; the expressions I find to be as follows:

---

* Clebsch, "Vorlesungen über Geometrie" (Lindemann), pp. 288–291 and note on p. 290.

$$\left.\begin{aligned} f &= U, \\ F_{11} &= 2\Theta_2, \\ u_x^2 A_{111} &= 6(\Theta_2 U - \Theta_4), \\ u_x^2 A_{112} &= 2(U'\Theta_2 - 2\Lambda_4 - \Psi_4 + UF_{12}), \\ u_x B_1 &= 2(U'\Theta_2 - \Lambda_4), \end{aligned}\right\}$$

$$\left.\begin{aligned} f' &= U', \\ F_{22} &= 2\Theta_2', \\ u_x^2 A_{222} &= 6(\Theta_2' U' - \Theta_4'), \\ u_x^2 A_{221} &= 2(U\Theta_2' - 2\mathrm{M}_4 - X_4 + U'F_{12}), \\ u_x B_2 &= 2(U\Theta_2' - \mathrm{M}_4), \end{aligned}\right\}$$

$$\left.\begin{aligned} N &= -\Phi, \\ u_x C_1 &= 2(G - \Theta_2\Phi), \\ u_x C_2 &= 2(G' - \Theta_2'\Phi), \\ u_x^2 D &= 4(\Theta_2 G' + \Theta_2' G - 2\Theta_2\Theta_2'\Phi - u_x F_{12} G_{12}), \end{aligned}\right\}$$

$$\left.\begin{aligned} N &= 4G_{12}, \\ u_x^2\Gamma_1 &= 4(\Phi\Lambda_4 - U'G + u_x U G_{12}), \\ u_x^2\Gamma_2 &= 4(\Phi\mathrm{M}_4 - UG' + u_x U' G_{12}), \\ u_x^3\Delta &= 4(\Phi^3 - U'\Phi\Psi_4 - U\Phi X_4 + U'\Phi\Lambda_4 + U\Phi M_4 \\ &\qquad - U^2 G' - U^2 G + UU'\Phi F_{12} + 2u_x UU' G_{12}), \end{aligned}\right\}$$

$$\left.\begin{aligned} f_{12} &= F_{12}, \\ u_x^2\Phi_{12} &= 4(\Phi^2 - UX_4 - U'\Psi_4 + UU'F_{12}), \end{aligned}\right\}$$

the symbols on the left-hand sides being those used by Gordan. From these relations it is easy to deduce the equations

$$\begin{aligned} u_x D &= F_{11} C_2 + F_{22} C_1 - N f_{12}, \\ u_x \Delta &= f'\Gamma_1 + f\Gamma_2 - N\Phi_{12}, \end{aligned}$$

subsisting among Gordan's concomitants.

VI.—*System of Three Quadratics.*

53. They may be taken in the forms

$$\begin{aligned} &a_0 x_1^2 + 2b_0 x_1 x_2 + 2a_1 x_1 x_3 + c_0 x_2^2 + 2b_1 x_2 x_3 + a_2 x_3^2, \\ &a_0' x_1^2 + 2b_0' x_1 x_2 + 2a_1' x_1 x_3 + c_0' x_2^2 + 2b_1' x_2 x_3 + a_2' x_3^2, \\ &a_0'' x_1^2 + 2b_0'' x_1 x_2 + 2a_1'' x_1 x_3 + c_0'' x_2^2 + 2b_1'' x_2 x_3 + a_2'' x_3^2; \end{aligned}$$

the characteristic equations are

$$D_1 + D_1' + D_1'' = 0 \text{ and } \Delta = D_6 + D_6' + D_6'' = 0.$$

There are eighteen coefficients in all; there will therefore be seventeen equations subsidiary to the first of the characteristic equations, requiring seventeen independent integrals. The number of modified Δ-equations is sixteen, and there will therefore be fifteen solutions independent of one another and common to the two characteristic equations.

Hence *all the simultaneous concomitants can be expressed in terms of fifteen concomitants.*

In what follows, only the results are given; they are derived by algebraical analysis similar to what has preceded, and the solutions evidently maintain the preceding analogy to binariants.

In forming the equations, the following quantities occur:

$$\left.\begin{aligned} \theta_0 &= a_0 \\ \theta_1 &= a_1 \\ \theta_2 &= a_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} \theta_0' &= a_0' \\ \theta_1' &= a_1' \\ \theta_2' &= a_2' \end{aligned}\right\}, \qquad \left.\begin{aligned} \theta_0'' &= a_0'' \\ \theta_1'' &= a_1'' \\ \theta_2'' &= a_2'' \end{aligned}\right\};$$

$$\left.\begin{aligned} \theta_3 &= b_1 a_1 - a_2 b_0 \\ \psi_3 &= b_1' a_1 - a_2' b_0 \\ \xi_3 &= b_1'' a_1 - a_2'' b_0 \end{aligned}\right\}, \qquad \left.\begin{aligned} \chi_3 &= b_1 a_1' - a_2 b_0' \\ \theta_3' &= b_1' a_1' - a_2' b_0' \\ \xi_3' &= b_1'' a_1' - a_2'' b_0' \end{aligned}\right\}, \qquad \left.\begin{aligned} \eta_3 &= b_1 a_1'' - a_2 b_0'' \\ \eta_3' &= b_1' a_1'' - a_2' b_0'' \\ \theta_3'' &= b_1'' a_1'' - a_2'' b_0'' \end{aligned}\right\}.$$

Of these quantities, only θ_0, θ_0', θ_0'' are solutions of the equations. The modified Δ-equations are constructed for the three possible cases, according as θ_1, θ_1', or θ_1'' is taken as the variable of reference.

The further quantities here following also occur; they all are simultaneous solutions of the two characteristic equations:

$$\left.\begin{aligned} \theta_4 &= (c_0,\ b_1,\ a_2 \between a_1,\ -b_0)^2 \\ \chi_4 &= (c_0,\ b_1,\ a_2 \between a_1',\ -b_0')^2 \\ \eta_4 &= (c_0,\ b_1,\ a_2 \between a_1'',\ -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \lambda_4 &= (c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between a_1',\ -b_0') \\ \lambda_4' &= (c_0,\ b_1,\ a_2 \between a_1,\ -b_0 \between a_1'',\ -b_0'') \\ \lambda_4'' &= (c_0,\ b_1,\ a_2 \between a_1',\ -b_0' \between a_1'',\ -b_0'') \end{aligned}\right\};$$

$$\left.\begin{aligned} \psi_4 &= (c_0',\ b_1',\ a_2' \between a_1,\ -b_0)^2 \\ \theta_4' &= (c_0',\ b_1',\ a_2' \between a_1',\ -b_0')^2 \\ \eta_4' &= (c_0',\ b_1',\ a_2' \between a_1'',\ -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \mu_4 &= (c_0',\ b_1',\ a_2' \between a_1,\ -b_0 \between a_1',\ -b_0') \\ \mu_4' &= (c_0',\ b_1',\ a_2' \between a_1,\ -b_0 \between a_1'',\ -b_0'') \\ \mu_4'' &= (c_0',\ b_1',\ a_2' \between a_1',\ -b_0' \between a_1'',\ -b_0'') \end{aligned}\right\};$$

$$\left.\begin{aligned} \xi_4 &= (c_0'',\ b_1'',\ a_2'' \between a_1,\ -b_0)^2 \\ \xi_4' &= (c_0'',\ b_1'',\ a_2'' \between a_1',\ -b_0')^2 \\ \theta_4'' &= (c_0'',\ b_1'',\ a_2'' \between a_1'',\ -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \nu_4 &= (c_0'',\ b_1'',\ a_2'' \between a_1,\ -b_0 \between a_1',\ -b_0') \\ \nu_4' &= (c_0'',\ b_1'',\ a_2'' \between a_1,\ -b_0 \between a_1'',\ -b_0'') \\ \nu_4'' &= (c_0'',\ b_1'',\ a_2'' \between a_1',\ -b_0' \between a_1'',\ -b_0'') \end{aligned}\right\};$$

where in the right-hand column λ_4 denotes $c_0 a_1 a_1' - b_1(a_1 b_0' + a_1' b_0) + a_2 b_0 b_0'$, and similarly for the others.

54. Taking first the equations independent of one another and formed with θ_1 as the variable of reference, we have

$$\left.\begin{array}{l}\theta_1\Delta\theta_2 - 2\theta_2\Delta\theta_1 = 2\theta_3 \\ \theta_1\Delta\theta_3 - \theta_3\Delta\theta_1 = \theta_4\end{array}\right\}, \quad \left.\begin{array}{l}\theta_1\Delta\theta_1' - \theta_1'\Delta\theta_1 = \phi \\ \theta_1\Delta\theta_2' - 2\theta_2'\Delta\theta_1 = 2\psi_3 \\ \theta_1\Delta\psi_3 - \psi_3\Delta\theta_1 = \psi_4\end{array}\right\}, \quad \left.\begin{array}{l}\theta_1\Delta\theta_1'' - \theta_1''\Delta\theta_1 = -\phi' \\ \theta_1\Delta\theta_2'' - 2\theta_2''\Delta\theta_1 = 2\xi_3 \\ \theta_1\Delta\xi_3 - \xi_3\Delta\theta_1 = \xi_4\end{array}\right\},$$

where $\phi = a_1 b_0' - a_1' b_0$, $\phi' = a_1'' b_0 - a_1 b_0''$ are simultaneous solutions of the two characteristic equations.

A set of independent solutions of these equations—necessarily seven in number to make up the required fifteen, for we already have θ_0, θ_0', θ_0''; θ_4, ψ_4, ξ_4; ϕ and ϕ'—is

$$\left.\begin{array}{l}(\theta_2\theta_4 - \theta_3^2) \div \theta_1^2 = \vartheta_2 = a_2 c_0 - b_1^2 \\ (\theta_2'\psi_4 - \psi_3^2) \div \theta_1^2 = \vartheta_2' = a_2' c_0' - b_1'^2 \\ (\theta_2''\xi_4 - \xi_3^2) \div \theta_1^2 = \vartheta_2'' = a_2'' c_0'' - b_1''^2\end{array}\right\},$$

$$\left.\begin{array}{l}(\theta_1'\theta_4 - \theta_3\phi) \div \theta_1 = \lambda_4 \\ (\theta_1'\psi_4 - \psi_3\phi) \div \theta_1 = \mu_4 \\ (\theta_1'\xi_4 - \xi_3\phi) \div \theta_1 = \nu_4\end{array}\right\},$$

$$(-\theta_1'\phi' - \theta_1''\phi) \div \theta_1 = \phi'',$$

where the quantities ϕ are

$$\left.\begin{array}{l}\phi = a_1 b_0' - a_1' b_0 \\ \phi' = a_1'' b_0 - a_1 b_0'' \\ \phi'' = a_1' b_0'' - a_1'' b_0'\end{array}\right\}.$$

Hence it follows that every simultaneous solution of the two equations can be expressed in terms of the fifteen independent solutions already obtained, viz. θ_0, θ_0', θ_0''; θ_4, ψ_4, ξ_4; ϕ, ϕ', ϕ''; ϑ_2, ϑ_2', ϑ_2''; λ_4, μ_4, ν_4.

As this set of fifteen is not symmetrical with regard to the three quadratics, it will be replaced immediately by an equivalent set of fifteen independent solutions which shall be symmetrical.

55. Taking now the equations for each of the three possible variables of reference, we find that the foregoing set of eight is increased when all the quantities which arise in the other sets are treated similarly with θ_1 as the variable; the new equations thus obtained are, of course, not independent equations as they can be derived from the eight, but it is convenient so to increase the set in

order to have them complete in form. Introducing quantities defined by the equations

$$\left.\begin{aligned} \theta_2 &= \theta_1^2 p = \theta_1'^2 r' = \theta_1''^2 t'' \\ \theta_2' &= \theta_1^2 r = \theta_1'^2 p' = \theta_1''^2 n'' \\ \theta_2'' &= \theta_1^2 t = \theta_1'^2 n' = \theta_1''^2 p'' \end{aligned}\right\}, \quad \left.\begin{aligned} \theta_3 &= \theta_1 q = \theta_1' s' = \theta_1'' l'' \\ \theta_3' &= \theta_1 s = \theta_1' q' = \theta_1'' k'' \\ \theta_3'' &= \theta_1 l = \theta_1' k = \theta_1'' q'' \end{aligned}\right\}, \quad \left.\begin{aligned} \theta_1 & = \theta_1'\varepsilon' = \theta_1''\gamma'' \\ \theta_1' &= \theta_1\varepsilon = \theta_1''\delta'' \\ \theta_1'' &= \theta_1\gamma = \theta_1'\delta' \end{aligned}\right\},$$

$$\left.\begin{aligned} \psi_3 &= \theta_1\rho = \theta_1'\omega' = \theta_1''\pi'' \\ \xi_3' &= \theta_1\omega = \theta_1'\rho' = \theta_1''\varkappa'' \\ \eta_3 &= \theta_1\pi = \theta_1'\varkappa' = \theta_1''\rho'' \end{aligned}\right\}, \quad \left.\begin{aligned} \chi_3 &= \theta_1\sigma = \theta_1'\iota' = \theta_1''\tau'' \\ \eta_3' &= \theta_1\iota = \theta_1'\sigma' = \theta_1''\upsilon'' \\ \xi_3 &= \theta_1\tau = \theta_1'\upsilon' = \theta_1''\sigma'' \end{aligned}\right\},$$

the three completed sets of modified Δ-equations are

$\theta_1^2\Delta = \nabla$	$\theta_1'^2\Delta = \nabla'$	$\theta_1''^2\Delta = \nabla''$
$\nabla p = 2q$	$\nabla' p' = 2q'$	$\nabla'' p'' = 2q''$
$\nabla q = \theta_4$	$\nabla' q' = \theta_4'$	$\nabla'' q'' = \theta_4''$
$\nabla \varepsilon = \phi$	$\nabla' \varepsilon' = -\phi$	$\nabla'' \gamma'' = \phi'$
$\nabla \gamma = -\phi'$	$\nabla' \delta = \phi''$	$\nabla'' \delta'' = -\phi''$
$\nabla r = 2\rho$	$\nabla' r' = 2\iota'$	$\nabla'' t'' = 2\rho''$
$\nabla \rho = \psi_4$	$\nabla' \iota' = \chi_4$	$\nabla'' \rho'' = \eta_4$
$\nabla t = 2\tau$	$\nabla' n' = 2\rho'$	$\nabla'' n'' = 2\upsilon''$
$\nabla \tau = \xi_4$	$\nabla' \rho' = \xi_4'$	$\nabla'' \upsilon'' = \eta_4'$
$\nabla \sigma = \lambda_4$	$\nabla' \varkappa' = \lambda_4''$	$\nabla'' \tau'' = \lambda_4''$
$\nabla \pi = \lambda_4'$	$\nabla' s' = \lambda_4$	$\nabla'' l'' = \lambda_4'$
$\nabla \iota = \mu_4'$	$\nabla' \omega' = \mu_4$	$\nabla'' \pi'' = \mu_4'$
$\nabla s = \mu_4$	$\nabla' \sigma' = \mu_4''$	$\nabla'' k'' = \mu_4''$
$\nabla \omega = \nu_4$	$\nabla' \upsilon' = \nu_4$	$\nabla'' \sigma'' = \nu_4'$
$\nabla l = \nu_4'$	$\nabla' k = \nu_4''$	$\nabla'' \varkappa'' = \nu_4''$

in each of which sets the first eight are the independent equations for that set.*

56. We first modify the algebraically complete system of solutions so that it may become symmetrical with regard to the three quantics. We have

$$\begin{aligned} \theta_4\chi_4 &= \lambda_4^2 \;\; + \phi^2\vartheta_2, \\ \theta_4\eta_4 &= \lambda_4'^2 \;\; + \phi'^2\vartheta_2, \\ -\;\lambda_4'\phi &= \lambda_4\phi' + \theta_4\phi'', \end{aligned}$$

so that χ_4 and η_4 may replace χ_4 and λ_4; and

$$\begin{aligned} \xi_4\xi_4' &= \nu_4^2 \;\; + \phi^2\vartheta_2'', \\ \psi_4\eta_4' &= \mu_4'^2 \;\; + \phi'^2\vartheta_2', \\ -\,\mu_4'\phi &= \mu_4\phi' + \psi_4\phi'', \end{aligned}$$

so that ξ_4' and η_4' may replace ν_4 and μ_4. Hence *all the simultaneous solutions can be expressed in terms of the algebraically complete set of fifteen constituted by* θ_0, θ_0', θ_0''; ϑ_2, ϑ_2', ϑ_2''; ϕ, ϕ', ϕ''; χ_4, η_4; ψ_4, η_4'; ξ_4, ξ_4', *a symmetrical set.*

The foregoing equations used for the modification of the system are selected from the following aggregate:

$$\left.\begin{aligned} \lambda_4'\phi + \lambda_4\phi' + \theta_4\phi'' &= 0 \\ \mu_4'\phi + \mu_4\phi' + \psi_4\phi'' &= 0 \\ \nu_4'\phi + \nu_4\phi' + \xi_4\phi'' &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} \lambda_4''\phi + \chi_4\phi' + \lambda_4\phi'' &= 0 \\ \mu_4''\phi + \theta_4'\phi' + \mu_4\phi'' &= 0 \\ \nu_4''\phi + \xi_4'\phi' + \nu_4\phi'' &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} \theta_4\chi_4 &= \lambda_4^2 + \phi^2\vartheta_2 \\ \psi_4\theta_4' &= \mu_4^2 + \phi^2\vartheta_2' \\ \xi_4\xi_4' &= \nu_4^2 + \phi^2\vartheta_2'' \end{aligned}\right\}, \quad \left.\begin{aligned} \theta_4\eta_4 &= \lambda_4'^2 + \phi'^2\vartheta_2 \\ \psi_2\eta_4' &= \mu_4'^2 + \phi'^2\vartheta_2' \\ \xi_4\theta_4'' &= \nu_4'^2 + \phi'^2\vartheta_2'' \end{aligned}\right\}, \quad \left.\begin{aligned} \chi_4\eta_4 &= \lambda_4''^2 + \phi''^2\vartheta_2 \\ \theta_4'\eta_4' &= \mu_4''^2 + \phi''^2\vartheta_2' \\ \xi_4'\theta_4'' &= \nu_4''^2 + \phi''^2\vartheta_2'' \end{aligned}\right\}.$$

57. As in the system of simultaneous concomitants for two quadratics, there are other solutions of the two characteristic equations (and so other concomitants) simple in form and useful because subsidiary to the expression of concomitants. The most important of these are:

---

*In the case of a system of n ternary quadratics, it is easy to see (1) that the number of equations in each of the n complete systems formed as above is $n^2 + 2n - 1$, and (2) that all the simultaneous concomitants are expressible in terms of $6n - 3$ concomitants properly chosen.

$$\left.\begin{aligned} f_{12} &= a_2c_0' + a_2'c_0 - 2b_1b_1' \\ f_{23} &= a_2'c_0'' + a_2''c_0' - 2b_1'b_1'' \\ f_{13} &= a_2c_0'' + a_2''c_0 - 2b_1b_1'' \end{aligned}\right\};$$

$$\left.\begin{aligned} g &= (A, B, C \between a_1, -b_0)^2 \\ g' &= (A, B, C \between a_1', -b_0')^2 \\ g'' &= (A, B, C \between a_1'', -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} g_{12} &= (A, B, C \between a_1, -b_0 \between a_1', -b_0') \\ g_{13} &= (A, B, C \between a_1, -b_0 \between a_1'', -b_0'') \\ g_{23} &= (A, B, C \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\},$$

$$\left.\begin{aligned} j &= (A', B', C' \between a_1, -b_0)^2 \\ j' &= (A', B', C' \between a_1', -b_0')^2 \\ j'' &= (A', B', C' \between a_1'', -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} j_{12} &= (A', B', C' \between a_1, -b_0 \between a_1', -b_0') \\ j_{13} &= (A', B', C' \between a_1, -b_0 \between a_1'', -b_0'') \\ j_{23} &= (A', B', C' \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\},$$

$$\left.\begin{aligned} e &= (A'', B'', C'' \between a_1, -b_0)^2 \\ e' &= (A'', B'', C'' \between a_1', -b_0')^2 \\ e'' &= (A'', B'', C'' \between a_1'', -b_0'')^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} e_{12} &= (A, B, C \between a_1, -b_0 \between a_1', -b_0') \\ e_{13} &= (A, B, C \between a_1, -b_0 \between a_1'', -b_0'') \\ e_{23} &= (A, B, C \between a_1', -b_0' \between a_1'', -b_0'') \end{aligned}\right\};$$

where

$$\begin{aligned} \tfrac{1}{2}A &= b_1'c_0 - b_1c_0', & B &= a_2'c_0 - a_2c_0', & \tfrac{1}{2}C &= a_2'b_1 - a_2b_1', \\ \tfrac{1}{2}A' &= b_1c_0'' - b_1''c_0, & B' &= a_2c_0'' - a_2''c_0, & \tfrac{1}{2}C' &= a_2b_1'' - a_2''b_1, \\ \tfrac{1}{2}A'' &= b_1''c_0' - b_1'c_0'', & B'' &= a_2''c_0' - a_2'c_0'', & \tfrac{1}{2}C'' &= a_2''b_1' - a_2'b_1''. \end{aligned}$$

And the equations which express these functions in terms of the system of §56 are of the forms

$$\left.\begin{aligned} \tfrac{1}{2}g\phi &= \lambda_4\psi_4 - \mu_4\theta_4 \\ \tfrac{1}{2}g'\phi &= \mu_4\chi_4 - \lambda_4\theta_4' \\ \tfrac{1}{2}g''\phi'' &= \mu_4''\eta_4 - \lambda_4''\eta_4' \end{aligned}\right\}, \qquad \left.\begin{aligned} g_{12}\psi_4 &= \mu_4 g + (\psi_4 f_{12} - 2\theta_4\vartheta_2')\phi \\ g_{13}\psi_4 &= \mu_4' g + (2\theta_4\vartheta_2' - \psi_4 f_{12})\phi' \\ g_{23}\theta_4' &= \nu_4'' g' + (\theta_4' f_{12} - 2\chi_4\vartheta_2')\phi'' \end{aligned}\right\}$$

with two similar sets; and

$$\left.\begin{aligned} \tfrac{1}{4}g^2 &= -\vartheta_2'\theta_4^2 + f_{12}\theta_4\psi_4 - \vartheta_2\psi_4^2 \\ \tfrac{1}{4}j^2 &= -\vartheta_2\xi_4^2 + f_{13}\theta_4\xi_4 - \vartheta_2''\theta_4^2 \\ \tfrac{1}{4}e^2 &= -\vartheta_2''\psi_4^2 + f_{23}\psi_4\xi_4 - \vartheta_2'\xi_4^2 \end{aligned}\right\}$$

being one of three sets.

58. The orders in the x-variables and the classes in the u-variables are as follow, being most easily obtained from the symbolical forms:

ORDER.	CLASS.	LEADING COEFFICIENT.
0	2	$\vartheta_2,\ \vartheta_2',\ \vartheta_2'';\ f_{12},\ f_{23},\ f_{13}.$
2	0	$\theta_0,\ \theta_0',\ \theta_0''.$
2	1	$\phi,\ \phi',\ \phi''.$
2	2	$\left.\begin{matrix}\theta_4, & \chi_4, & \eta_4 \\ \lambda_4, & \lambda_4', & \lambda_4''\end{matrix}\right\};\ \left.\begin{matrix}\psi_4, & \theta_4', & \eta_4' \\ \mu_4, & \mu_4', & \mu_4''\end{matrix}\right\};\ \left.\begin{matrix}\xi_4, & \xi_4', & \theta_4'' \\ \nu_4, & \nu_4', & \nu_4''\end{matrix}\right\}.$
2	3	$\left.\begin{matrix}g, & g', & g'' \\ g_{12}, & g_{23}, & g_{13}\end{matrix}\right\};\ \left.\begin{matrix}j, & j', & j'' \\ j_{12}, & j_{23}, & j_{13}\end{matrix}\right\};\ \left.\begin{matrix}e, & e', & e'' \\ e_{12}, & e_{23}, & e_{13}\end{matrix}\right\}.$

which is to be read: that the concomitant in ϑ_2 as its leading coefficient is of order 0 and class 2, so that its first term is $\vartheta_2 u_1^2$, and so on.

All the simultaneous concomitants can be expressed in terms of the fifteen, which constitute the symmetrical set given by

$$\left.\begin{aligned} U &= \theta_0 x_1^2 + \dots \\ U' &= \theta_0' x_1^2 + \dots \\ U'' &= \theta_0'' x_1^2 + \dots \end{aligned}\right\},\quad \left.\begin{aligned} \Theta_2 &= \vartheta_2 u_1^2 + \dots \\ \Theta_2' &= \vartheta_2' u_1^2 + \dots \\ \Theta_2'' &= \vartheta_2'' u_1^2 + \dots \end{aligned}\right\},\quad \left.\begin{aligned} \Phi &= \phi x_1^2 u_1 + \dots \\ \Phi' &= \phi' x_1^2 u_1 + \dots \\ \Phi'' &= \phi'' x_1^2 u_1 + \dots \end{aligned}\right\},$$

$$\left.\begin{aligned} \mathrm{X}_4 &= \chi_4 x_1^2 u_1^2 + \dots \\ H_4 &= \eta_4 x_1^2 u_1^2 + \dots \end{aligned}\right\},\quad \left.\begin{aligned} \Psi_4 &= \psi_4 x_1^2 u_1^2 + \dots \\ H_4' &= \eta_4' x_1^2 u_1^2 + \dots \end{aligned}\right\},\quad \left.\begin{aligned} \Xi_4 &= \xi_4 x_1^2 u_1^2 + \dots \\ \Xi_4' &= \xi_4' x_1^2 u_1^2 + \dots \end{aligned}\right\};$$

the symbolical expressions for which are

$$U = \alpha_x^2, \qquad U' = \alpha_x'^2, \qquad U'' = \alpha_x''^2;$$

$$\Theta_2 = \tfrac{1}{2}(\alpha\beta u)^2, \qquad \Theta_2' = \tfrac{1}{2}(\alpha'\beta' u)^2, \qquad \Theta_2'' = \tfrac{1}{2}(\alpha''\beta'' u)^2;$$

$$\Phi = \alpha_x \alpha_x' (\alpha'\alpha u), \qquad \Phi' = \alpha_x''\alpha_x(\alpha\alpha'' u), \qquad \Phi'' = \alpha_x'\alpha_x''\, \alpha''\alpha' u);$$

$$\mathrm{X}_4 = \beta_x'\gamma_x'(\alpha\beta' u)(\alpha\gamma' u), \qquad H_4 = \beta_x''\gamma_x''(\alpha\beta'' u)(\alpha\gamma'' u);$$

$$\Psi_4 = \beta_x\gamma_x(\alpha'\beta u)(\alpha'\gamma u), \qquad H_4' = \beta_x''\gamma_x''(\alpha'\beta'' u)(\alpha'\gamma'' u);$$

$$\Xi_4 = \beta_x\gamma_x(\alpha''\beta u)(\alpha''\gamma u), \qquad \Xi_4' = \beta_x'\gamma_x'(\alpha''\beta' u)(\alpha''\gamma' u).$$

And for the purposes of expression, the remainder of the set of concomitants determined by the preceding table will be useful.

Thus the Jacobian (Salmon's Conic Sections, §388) is

$$-\frac{1}{u_x}(U\Phi'' + U'\Phi' + U''\Phi);$$

the involutant (ib. §388a) is

$$\frac{1}{8}\frac{GJE}{\Theta_4\Psi_4\Xi_4} - \frac{1}{2}\left(\frac{\Theta_2 E}{\Theta_4} + \frac{\Theta_2' J}{\Psi_4} + \frac{\Theta_2'' G}{\Xi_4}\right);$$

of the ten simultaneous invariants, which are asyzygetic, nine are given by equations similar to those which give the four asyzygetic invariants of two quadratics in §52; and the tenth, being

$$\Sigma a_0(a_2'c_0'' + a_2''c_0' - 2b_1'b_1'') - 2\Sigma\{c_0a_1'a_1'' - b_1(a_1'b_0'' + a_1''b_0') + a_2b_0'b_0''\},$$

is equal to

$$\frac{1}{u_x^2}\{UF_{23} + U'F_{13} + U''F_{12} - 2\Lambda_4'' - 2M_4'' - 2N_4''\}.$$

Similarly for other examples.

Some investigations dealing with a system of three quadratics are given by Cayley and Hermite in the 57th volume of Crelle's Journal, and by Gundelfinger in the 80th volume.

(*To be continued.*)